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C.J. WOODWARD'S
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Works.

ARITHMETICAL PHYSICS.

PART Ib.

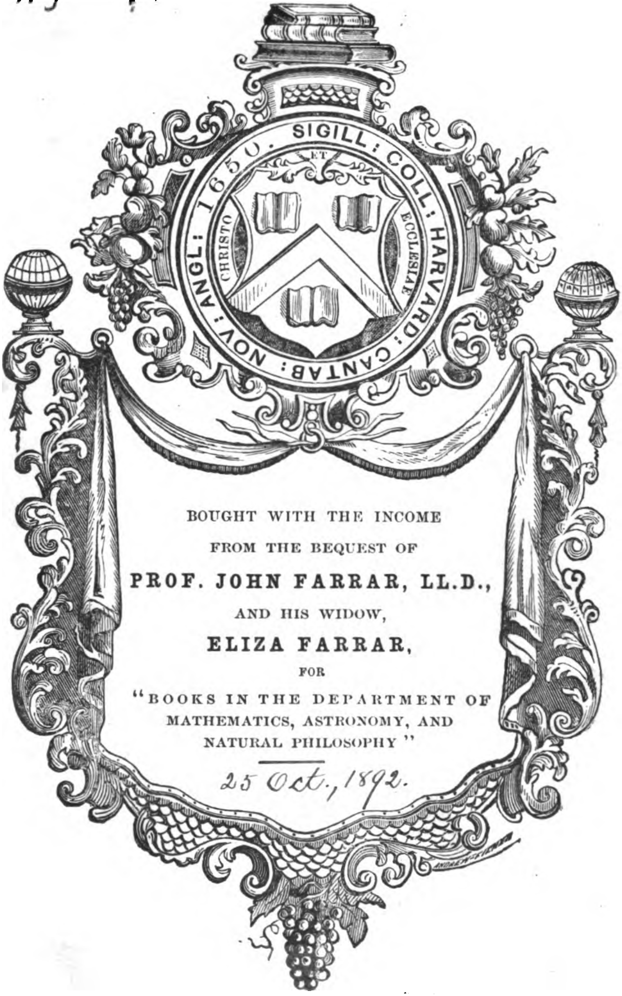
3/6

TRIGONOMETRICAL FUNCTIONS.

Ang.	Sine.	Cosine.	Tang.
0°	0·000	1·000	0·000
1	·017	1·000	·017
2	·035	0·999	·035
3	·052	·999	·052
4	·070	·998	·070
5°	0·087	0·996	0·087
6	·105	·995	·105
7	·122	·993	·123
8	·139	·990	·141
9	·156	·988	·158
10°	0·174	0·985	0·176
11	·191	·982	·194
12	·208	·978	·213
13	·225	·974	·231
14	·242	·970	·249
15°	0·259	0·966	0·268
16	·276	·961	·287
17	·292	·956	·306
18	·309	·951	·325
19	·326	·946	·344
20°	0·342	0·940	0·364
21	·358	·934	·384
22	·375	·927	·404
23	·391	·921	·424
24	·407	·914	·445
25°	0·423	0·906	0·466
26	·438	·899	·488
27	·454	·891	·510
28	·469	·883	·532
29	·485	·875	·554
30°	0·500	0 866	0·577
31	·515	·857	·601
32	·530	·848	·625
[illegible]	[illegible]	[illegible]	·649
[illegible]	[illegible]	[illegible]	675

Ang.	Sine.	Cosine.	Tang.
35°	0·574	0·819	0 700
36	·588	·809	·727
37	·602	·799	·754
38	·616	·788	·781
39	·629	·777	·810
40°	0·643	0·766	0·839
41	·656	·755	·869
42	·669	·743	·900
43	·682	·731	·933
44	·695	·719	·966
45°	0·707	0·707	1·000
46	·719	·695	·036
47	·731	·682	·072
48	·743	·669	·111
49	·755	·656	·150
50°	0·766	0·643	1·192
51	·777	·629	·235
52	·788	·616	·280
53	·799	·602	·327
54	·809	·588	·376
55°	0·819	0·574	1·428
56	·829	·559	·483
57	·839	·545	·540
58	·848	·530	·600
59	·857	·515	·664
60°	0·866	0·500	1·732
61	·875	·485	·804
62	·883	·469	·881
63	·891	·454	·963
64	·899	·438	2·050
65°	0·906	0·423	2·145
66	·914	·407	·246
67	·921	·391	·356
68	·927	·375	·475
69	·934	·358	·605

Phys 247.13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OR,

ARITHMETICAL PHYSICS,

PART IB.

ACOUSTICS, LIGHT, AND HEAT,

(DEGREE AND HONOURS STAGES,)

BY

C. J. WOODWARD, B.Sc.,

PRINCIPAL, CHEMISTRY AND PHYSICS DEPARTMENT,
BIRMINGHAM AND MIDLAND INSTITUTE,
BIRMINGHAM.

LONDON:
SIMPKIN, MARSHALL & CO.
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GENERAL PREFACE

To Arithmetical Chemistry and Arithmetical Physics.

THE series of books with the above titles originated in a collection of exercises that were prepared by the author, and published many years ago.

From mere exercises with illustrative examples these books, while retaining their individuality, have become more and more converted into text-books. The plan of each is the same. A series of descriptive lessons is given, and each lesson contains illustrative examples fully worked out. By means of carefully graduated questions, the student is exercised in the phenomena and laws which have been treated in the lessons, while at the end of each book is a miscellaneous collection of questions (with answers appended) selected from the papers of various examining bodies.

I believe the method of teaching adopted in these books to be a sound one. Some critics feel alarmed when a book has examination questions in it; but just as fire is a good servant or a bad master, so examination questions are good or bad, according to whether we keep them subordinate or allow them to take first place. To a conscientious teacher nothing can be sadder than to see a student who, with muddled mind and half-knowledge, pores over old examination papers, picking up rules here and there to help him in their solution. On the other hand, a student who has studied the first principles of a science, and merely requires exercise in the application of these first principles to obtain a thorough mastery of them, is greatly benefited by working through a series of questions which have been framed, as examination questions are, or should be, solely to find out whether these first principles are understood or not.

In using these books, then, the student should

1. Study the descriptive lessons.
2. Work through the exercises at end of each lesson.
3. Work through the examination questions at end of the book.

C. J. WOODWARD.

Birmingham and Midland Institute,
Birmingham, August, 1890.

PREFACE TO BOOK D.

ARITHMETICAL PHYSICS, PART IB.

THIS book, though fourth of the series. is published some time after its nominal successors have appeared. Many circumstances have hindered its production, the principal delay being due to a desire on the part of the author to make the work as complete and satisfactory as possible. This desire has led to the manuscript being frequently revised, and the opinions of several friends obtained upon it.

The lessons in this book are followed by Reference Chapters, which will be found useful. The last one on "General Phenomena of Wave Interference" is of great importance, and should be very carefully studied.

In several cases a departure has been made from the routine method of treatment, more especially in Lesson II. (Nodes and Ventral Segments), Lesson IV. (Inclined Mirrors), Lesson V. (Media with Parallel Sides), Lesson VIII. (Newton's Rings), Lesson X. (Hygrometry), and Lesson XV. (Kinetic Theory of Gases).

I am indebted to Mr. A. E. Briscoe for help in reading proofs, and also for writing several notes.

C. J. WOODWARD.

Birmingham and Midland Institute,
Birmingham, August, 1890.

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ARITHMETICAL PHYSICS.

PART IB.

LESSON I.

Elasticity, Density, Young's Modulus.

UNLESS the student is already familiar with the mode of measuring force, energy, and work in absolute units, he is strongly advised to read through Reference Chapter B, near the end of the book.

As stated on page 5 of *Arithmetical Physics, IA*, elasticity is measured by the pressure required to compress (or dilate) a substance. To compare elasticities, then, we merely compare the pressures required to compress (or dilate) substances to the same extent. The numbers given under "Elasticity" on page 5 of *Arithmetical Physics, IA*, are comparative numbers.

The actual elasticity of a liquid is the pressure which would reduce unit volume of the liquid to *nothing*, on the supposition that diminution would go on regularly and indefinitely with increase of pressure. Thus the pressure in pounds per square foot that would reduce a cubic foot of water to nothing is the elasticity of water in poundals; or the pressure in dynes per square centimetre that would reduce a cubic centimetre of water to nothing is the elasticity of water in dynes.

It would be absurd to expect that we could carry out by experiment the compressibility of a liquid to the extent implied in the above paragraph, for long before we had reduced the liquid to small dimensions the properties of the body would have changed. We may, however, easily determine the compressibility through a small range, and then assuming that the same relation would hold indefinitely, we can find the elasticity. Imagine a cubic centimetre of a substance of such material that when a pressure of one dyne is applied to its surface it shrinks 1/1000th of its volume, that if two dynes be applied it shrinks 2/1000ths of its volume, if three, then 3/1000ths, and so on. Evidently *if this compression goes on at the same rate*, then, if 1000 dynes be applied, the substance will have no volume, and this (1000 dynes) is the elasticity of the substance in C.G.S. measure. Using British measure, if a cubic foot of a material were diminished 1/1000th of a cubic foot by a pressure of one poundal per square foot, then 1000 poundals pressure would reduce the cubic foot to nothing, and 1000 poundals would be the elasticity of the material.

EXAMPLE A. Colladon and Sturm (*Ann. Chim. Phys.*, 1827, [2]. xxxvi., 113), (Miller's *Chemical Physics*, 1877, p. 49), found that for a range of twenty-four atmospheres, water at 0°C. diminished in volume 51·3 millionths of its original volume for each additional atmosphere (1·033 kilogrammes per sq. cm.). Required the elasticity of water at 0°C., and under atmospheric pressure.

Since a pressure of 1·033 kilogrammes, that is, 1033 (grammes) × 981 or 1,013,373 dynes, will reduce one million parts of water 51·3 parts, twice the pressure will reduce it 102·6 parts, ten times the pressure 513 parts, and 1,000,000/51·3, or 19,493 times the pressure, will on supposition reduce it to 0; that is, a pressure of 1,013,373 × 19,493, or 19,753,679,889 dynes, will reduce it to nothing; that is, the elasticity is 19,753,679,889 dynes, or $1·975368 \times 10^{10}$ very nearly.

With solids, the coefficient of resilience of volume cannot be obtained directly, inasmuch as uniform pressure cannot well be applied to the whole surface of a solid as it can be in the case of a liquid. With solids the *longitudinal resilience* or *Young's Modulus of Elasticity* is experimentally determined by straining a wire of the material, then adding a weight, and finding to what extent the wire stretches; from which the modulus is calculated by determining what pressure would double the length of the wire, supposing it to have an area of one square centimetre.

EXAMPLE B. A wire one square millimetre in area was strained by a weight so as to remove all bends. A weight of one kilogramme was hung on the wire, when it was found that it had stretched 1/1000th of its length; another kilogramme caused it to stretch another 1/1000th of its original length. Required the value of Young's modulus for the material of the wire.

If the wire had been one sq. cm. area, then 100 kilogrammes, or 100,000 grammes, would have been required to stretch the wire 1/1000th of its length, 200 kilogrammes, or 200,000 grammes, would have been required to stretch it 2/1000ths of its length, and 100,000 kilogrammes, or 100,000,000 grammes, would have been required to stretch it 1000/1000ths of its length, that is, to have doubled it in length. If the value of gravity is 981 dynes at the place of experiment, then pressure required to double the length of one centimetre of the wire of one sq. cm. area (on hypothesis that lengthening goes on equally) is 100,000,000 × 981, or 98,100.000,000 dynes, and this is the value of Young's modulus for the material of the wire.

The modulus may also be obtained by determining to what extent a pressure will reduce a rod of one square centimetre area, thus:—If a rod of one sq. cm. area and ten cm. length is reduced 1/1000th of its length when a weight of one kilogramme, that is, a pressure of 981,000 dynes, is applied, a thousand times this quantity, that is, 981,000,000 dynes, would reduce the rod to nothing, and this is the modulus. The modulus may be determined according to convenience, either from lengthening or shortening of the rod by a force applied.

A curious result follows if we consider a rod of the material giving a pressure equal to its modulus. Suppose a rod of the material of the above illustrations, one sq. cm. area of section, and weighing 100,000 grammes, raised upright, the pressure at its base would be 981,000,000 dynes; if the rod were put on a cubic centimetre of its own material, this would be crushed to nothing. Again, if a cubic centimetre of the material were put on the top of the rod, the additional weight would be just sufficient to crush to nothing a cubic centimetre at the base, and so, however the rod might be increased at the top, it would be reduced at the bottom to the same extent. (See Young's *Natural Philosophy*, Kelland, 1845, p. 106.)

Work is measured in lengthening a wire by the product of the average resistance into the space through which the resistance is overcome, thus on stretching a wire which already has a weight of one kilogramme upon it by a weight of ten kilogrammes, which stretches the wire through three centimetres, the average resistance is ½ (1000 + 10,000)g, or 5500g dynes, and this multiplied by three gives 16,500g ergs as the work done. The student may with advantage apply the indicator diagram (see Reference Chapter A) to find the work done, bearing in mind that pressure supported is the measure of resistance.

It has, no doubt, been noticed that in determining the value of Young's modulus the pressure or strain has to be reduced to unit area. Thus, if F is the tension in dynes on a wire of a square cm. area, and its length is changed by this tension from L to $L + l$, then the modulus e will be

$$e = \frac{F}{a} \div \frac{l}{L} = \frac{F}{a} \cdot \frac{L}{l}$$

and if T is the tension required to double the length of any wire, that is, make $l = L$, then

$$e = \frac{T}{a} = \frac{T}{\pi r^2} \qquad \text{(i.)}$$

where r is radius of the wire, because $a = \pi r^2$.

It will be noticed from a study of Example A, page 2, that to obtain the elasticity of the water we apply a small pressure, and note the voluminal compression produced, that is, note (i.) pressure applied, and (ii.) what shrinking takes place as compared to the original volume.

The ratio $\dfrac{\text{Shrinking}}{\text{Original volume}}$ is called the *voluminal compression.*

$$e = \frac{\text{Pressure applied}}{\dfrac{\text{Shrinking}}{\text{Original volume}}} = \frac{\text{Small increment of pressure}}{\text{Voluminal compression}}$$

or

$$e = \frac{P}{\dfrac{v}{V}} \qquad \text{(ii.)}$$

where P is the increment of pressure, v the shrinking, and V the original volume, and this is the expression for the elasticity of a fluid, as given by Clerk Maxwell, *Theory of Heat*, p. 107, where we read—"*The elasticity of a fluid under any given conditions is the ratio of any small increase of pressure to the voluminal compression hereby produced.*"

Elasticity of Gases. In applying the method of Example A, or, what comes to the same thing, the formula ii., to the calculation of the elasticity of a gas, we are beset by the difficulty that every increment of pressure increases the elasticity of the gas; if, however, the increment of pressure be *very small*, we may show by the following reasoning that the elasticity of a gas is equal to the pressure it supports.

Consider a volume V of a gas under a pressure P. Let a small increment of pressure p be applied, which reduces the volume V to a volume $V-v$. Then by Boyle's law, which states that the pressure on a gas multiplied into its volume is always constant if the temperature remain the same, we have

$$(P+p)(V-v)=PV \qquad (a)$$

or

$$PV+pV-Pv-pv=PV$$

or

$$pV-Pv=pv$$

Now pv being product of quantities extremely small as compared with P and V, we may assume

$$pV-Pv=0, \text{ or } pV=Pv$$

from which

$$P=\frac{pV}{v}=\frac{p}{\frac{v}{V}}$$

But P is the pressure which the gas originally supported, and $p/v/V$ is the elasticity of the gas (ii.) Hence the elasticity of a gas is the pressure which it supports, or

$$e=P$$

For this method of proof I am indebted to Mr. A. E. Briscoe, who has supplied it from notes of Professor Rücker's lectures.*

---

* The same result may also be obtained thus:—

$$(P+p)(V-v)=PV \qquad \text{Boyle's Law.}$$

or

$$\frac{P+p}{P}=\frac{V}{V-v}$$

But $\frac{P+p}{P}=1+\frac{p}{P}$ and $\frac{V}{V-v}=1+\frac{v}{V}+\left(\frac{v}{V}\right)^2+\left(\frac{v}{V}\right)^3$, &c., as proved in Algebra, *Geometrical Series*.

Hence neglecting $\left(\frac{v}{V}\right)$, &c., we have

$$1+\frac{p}{P}=1+\frac{v}{V} \text{ or } P=\frac{p}{\frac{v}{V}} \text{ as before.}$$

By removing a small pressure p from a volume V of air confined under a large pressure P we may obtain the same result, for then in the formula (a) the only difference is that p is *minus* and v *plus*, and this brings out the same result as before.

If the air undergoing compression or dilatation be confined in a vessel impervious to heat, or if, as in the transmission of a sound wave, the compressions and rarefactions are so rapid that the air has not time either to part with heat while the compression lasts, or to absorb heat while the rarefaction lasts, the elasticity is no longer simply P, but $1{\cdot}41P$ in air and most gases. The way in which the constant 1·41 is obtained will be referred to in Lesson XII. Specific Heat of Air, &c.; but it may here be pointed out how both the condensation and rarefaction increase the elasticity of the gas.

The condensation is very simple, for the heat developed by condensation evidently increases the power of the gas to support pressure, or in order to produce a small compression p must be greater.

Again, to produce a small rarefaction, p must be greater when the air is chilled than when it is at constant temperature, and as the coefficient of elasticity, that is, $p/v/V$, varies directly, as the magnitude of p, we see that in both condensation and rarefaction the elasticity is increased, because p is greater in both cases.

It may be mentioned here that the increase of elasticity due to the effect of the sound wave itself is simply the ratio of the specific heat of the gas at constant pressure to the specific heat of the gas at constant volume. This ratio is expressed variously as $\frac{Cp}{Cv}$, k or γ. See Lesson XII.

Homogeneous Atmosphere. The velocity of sound in a gas $= \sqrt{\frac{e}{d}}$ where e is the elasticity and d the density of the gas, but the velocity of sound may be given as *the velocity that a body falling freely would acquire in falling from a height equal to that of one-half the height of the homogeneous atmosphere.* (Newton's formula.)

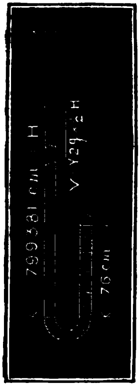

Fig. 1.

Let the J shaped barometer, fig. 1, have mercury in the shorter (closed) limb, while the longer limb is filled with air of uniform density throughout, the density corresponding to that at normal pressure and temperature. Suppose the bore of the tube to be one square centimetre, and that the height of the mercury column is 76 centimetres, and that the temperature of mercury and air is 0°C. Now, assuming gravity normal, the mass of the mercury is $76 \times 13{\cdot}6$, or 1033·6 grammes, and the mass of one cubic centimetre of air is 0·001293 gm. Hence there will be as many cubic centimetres of air in the tube as there are 0·001293 gm. in 1033·6 gm., that is, 799,381. So the height of the homogeneous atmosphere when gravity is normal is 799,381 centimetres; the half of this is 399,690·5, and the velocity of sound in air. according to Newton's formula, is the velocity a body would acquire in falling 399,690·5 centimetres. and as $v=\sqrt{2gs}$, taking g as 981, we have for the velocity of sound

$$v=\sqrt{2\times 981\times 399{,}690{\cdot}5}=28{,}003 \text{ cm.}$$

When this number is corrected for increase of elasticity by passage of the sound wave, that is, multiplied into $\sqrt{1{\cdot}41}$, the velocity coincides with that determined by experiment.

EXAMPLE C. The density of carbon dioxide that is mass of one c cm. when gravity is normal is 0·001977 gm. at 0°C., and 76 cm. mercurial pressure. Find the height of the homogeneous atmosphere of carbon dioxide under 70 cm. mercurial pressure, and at a temperature of 20°C.

Using the illustration of fig. 1, the mass of the carbon dioxide is of course equal to that of 70 c.cm. of mercury at 20°C. Without sensible error this may be assumed the same as at 0°; hence mass of the carbon dioxide in the open limb of tube is $70 \times 13{\cdot}6$, or 952 grammes, and the mass of one cubic centimetre of the carbon dioxide will be

$$0{\cdot}001977 \times \frac{70}{76} \times \frac{273}{293}, \text{ or } 0{\cdot}0016966$$

and the height in centimetres of the homogeneous atmosphere will be 952 gm. divided by 0·0016966 gm. = 561,113 cm.

EXAMPLE D. Find the velocity of sound in carbon dioxide under a pressure of 70 cm. of mercury and at a temperature of 20°C., taking the ratio of specific heats as 1·265 (Clausius).

From Example B we find one-half the height of the homogeneous atmosphere as 280,556·5 cm.; hence the velocity of sound in centimetres per second

$$=\sqrt{2\times 981\times 280{,}556{\cdot}5\times 1{\cdot}265}$$

or 26,388 cm. per second.

NOTE.—The ordinary formula $v=\sqrt{\frac{e}{d}}$ leads to the same result. For e as we have seen is the pressure that the carbon dioxide supports, viz., 70 c.cm. of mercury $= 70\times 13{\cdot}6\times 981\times 1{\cdot}265$ grammes, while $d=0{\cdot}001977\times\frac{70}{76}\times\frac{273}{293}$

EXERCISE I.

QUESTIONS WITH IMAGINARY CONSTANTS.

These can be worked without trouble by ordinary arithmetic.

1. A cubic centimetre of a fluid when submitted to a pressure of 100 grammes per square centimetre was reduced in volume by one cubic millimetre. Required (i.) its compressibility per 1000 dynes pressure; (ii.) the coefficient of volume resilience, that is, elasticity of the fluid. Take $g = 1000$ dynes.

2. What is the elasticity (Young's modulus) of a material when a wire of this material one square millimetre in area is found to stretch one-hundredth of its length on submitting the stretched wire to a pressure of 1,000,000 dynes?

3. (i.) What is the velocity of sound in the fluid of Question 1, supposing its density to be 4? (ii.) What is the velocity of sound in the wire of Question 2, if the mass of a cubic centimetre of the wire is 9 grammes?

4. What is the height of the homogeneous atmosphere of a gas, one cubic centimetre of which has a mass of 0·01 gramme at 0°C., and 100 cm. mercurial pressure, supposing the atmosphere to be at 0°C., and to be balanced by a column of mercury 100 centimetres high, and of density 13·6?

5. What is the velocity of sound in the gas of Question 4? Take $g=1000$, and ratio of specific heats as 1·44.

6. A cubic foot of a liquid diminishes by one cubic inch when a pressure of 144,000 lb. per square foot is applied at a place where $g=32$. The mass of one cubic foot of the liquid is 100 lb. Required (i.) the compressibility of the liquid for the pressure applied (144,000 × 32 poundals); (ii.) the elasticity of the liquid in poundals; (iii.) the velocity of sound in the liquid.

EXERCISES INVOLVING MORE DIFFICULT ARITHMETICAL OPERATIONS.

These should be worked with the help of Logarithms.

7. The specific heat of air at constant pressure C_p is 0·2375, and at constant volume C_v it is 0·1684. (i.) What is the ratio of these two specific heats? With ether vapour $C_p = 0{\cdot}4797$, and $C_v = 0{\cdot}453$. (ii.) What is the ratio in this case?

(iii.) Determine the velocity of sound in air at a temperature of 273°C., under a pressure of 76 cm. of mercury, or 1,013,961·6 dynes. Density of air at 0° and 76 cm. = 0·001293 gm.

(iv.) Determine the velocity of sound in ether vapour at a temperature of 273°C. and 76 cm. mercurial pressure. Density of ether vapour (assumed) at 0° and 76 cm. = 0·0033 gm.

8. A wire one square millimetre in cross section is straightened by hanging upon it a weight of one kilogramme It is now found that on hanging 98·1 kilogrammes on the wire it stretches 1/10000th of its length. Required the elasticity (Young's modulus) of the wire. Take $g = 981$.

9. The density of the above wire was found to be 9 grammes per cubic centimetre. Required the velocity of sound along the wire. (The velocity of sound along the wire must not be confused with the lateral pulse which may be conveyed along the wire.)

10. A cubic metre of water when compressed by a pressure of one million dynes (a megadyne) per square centimetre becomes reduced in volume by 50 cubic centimetres. Required the elasticity of water (coefficient of volume resilience).

11. The compressibility of ether at 0°C. is given as $1{\cdot}09 \times 10^{-4}$ for one megadyne pressure, while its density is about 0·736. From these data calculate the velocity of sound in liquid ether.

12. Calculate the height of the homogeneous atmosphere of each of the following gases, when the barometer reduced to 0°C. stands at 72 cm., and the temperature is 20°C.: air, hydrogen, carbon dioxide. (Density at 0°C. and 76 cm. pressure of air = 0·001293, hydrogen 0·0000896, carbon dioxide 0·00197)

13. From data of Question 12 determine velocity of sound at 0°C. in each of the gases by Newton's formula, that is, neglecting the ratio of specific heat at constant pressure to heat at constant volume.

ANSWERS.

1. (i.) 1/100000th, or 10^{-5}; (ii.) 100,000,000, or 10^8.

2. 10^{10}.

3. (i) 500 cm.; (ii.) $3{\cdot}3 \times 10^4$.

4. 136,000 cm.

5. 139,944 cm. per second.

6. (i.) 1/1728; (ii.)7·962624000; (iii.) 17,746,784 feet per second.

7. (i.) 1·410; (ii.) 1·059 nearly; (iii.) 47,025 cm.; (iv.) 25,510 cm.

8. $9{\cdot}62361 \times 10^{13}$.

9. 3,270,000 cm. per second.

10. 2×10^{10}.

11. 111,650 cm. per second.

12. Air, $8{\cdot}5793 \times 10^5$ cm.; hydrogen, $14{\cdot}43 \times 8{\cdot}5793 \times 10^5$ cm.; carbon dioxide, $0{\cdot}6563 \times 8{\cdot}579 \times 10^5$ cm.

13. Air, 29,011 cm.; hydrogen, 107,690 cm. per second; carbon dioxide, 23,503 cm. per second.

LESSON II.

Nodes and Ventral Segments. Laws of Vibrations of Strings.

Nodes and Ventral Segments. Following the principles stated in Lesson I. of *Arithmetical Physics, Part IA*, let a series of impulses be given to a string so as to produce a wave in the string. The simplest case is that of *simple harmonic motion*, where the impulses or energies producing the wave are such as those represented in fig. 1, plate 3, impulses which, starting from zero, vary continuously in strength in the positive and negative direction at the same rate as the chords of a circle vary.

Plate 3. Let the energies of fig. 1, plate 3, be applied in succession at the extreme end A of the string AB, fig. 2, plate 3. Suppose each impulse to take one second to travel along the length AB, and that the series of energies 0_1 1 2 3 ...... 9 0_2 $\bar{9}$ $\bar{8}$ ...... $\bar{2}$ $\bar{1}$ 0 are produced in one second. Such a result would be obtained experimentally by taking a string AB with one end B fastened to a support, and the end A fastened to one prong of a tuning fork, the time of vibration of the fork being one second, and the time taken for each impulse to travel along the string being also one second. It is also perhaps necessary to state that the plane of vibration of the fork should be at right angles to the line of the string when at rest.

When the series of impulses 0_1 to 1 have been communicated at A, they travel towards B, and are there reflected in the negative direction. The impulses 1 – 2 follow these, and are reflected, and so on throughout the positive series The negative series $0_2 - \bar{9}$, $\bar{9} - \bar{8}$, &c., follow on, and these are reflected in the positive direction. Hence if we fix our attention in the first place on the direct wave only, we can easily see that at the end of one second the impulses 0 – 1, fig. 2, will be at the end B, while the impulses $\bar{1} - 0$ will just have been communicated to A, and the string will have taken the wave-form of the dotted line in fig. 2. Let us now consider the reflected wave, which begins to be formed immediately the series $0_1 - 1$ reaches *B*. The various impulses $0 - 1$, $1 - 2$, &c., as they reach *B* are reflected in the reverse direction, and in one second after the wave of fig. 2 has been formed all the impulses producing the wave have travelled up to *B*, and have been reflected as shown in fig. 3, and these produce a wave precisely similar to the direct wave, but travelling in the opposite direction.

Suppose now the impulses continued at *A*, so that both direct and reflected waves are produced on the cord together. What effect will be produced?

I have found the following mechanical contrivance help to clear up this point, which is always one of difficulty to a beginner.

Take a strap 12 feet long, make it into an endless band, and put this band over pulleys, so that it takes the form represented in fig. 4 of plate 3; the pulleys themselves are not represented for the sake of clearness. Put cardboard arrows on the strap, representing four waves, each three feet long, A_1B_1 one wave, B_1B_2 a second one, B_2A_2 the third wave, and A_2A_1 the fourth wave. Screen from view all above the line B_1B_2, and all below the line A_1A_2; then the two exposed waves will represent—one the direct, the other the reflected wave; and when the pulleys are turned, we can see by looking at the arrows which are opposite each other what is the effect of the interference produced.

Let the pulleys be turned so as to make the strap move in the direction of the vertical arrows, and it will easily be seen that as the small arrows, 1, 2, 3, &c., of the direct wave A_1B_1 pass behind the upper screen, the arrows $\bar{1}$, $\bar{2}$, $\bar{3}$, &c., appear in front, forming the reflected wave, and it is obvious that at the line B_1B_2 the arrows are always equal and opposite in direction as the pulleys turn. The part B_1B_2, representing one end of the string, will then be a node, similarly the part A_1A_2 will be a node, and also the middle of the string *N*. Consider now those parts marked *V*. As the direct and reflected waves pass each other at these points it will be noticed that the arrows are always of equal length and in the same direction, while at points intermediate between *V* and *N* intermediate effects are produced.

A still better way to enable the student to realise easily the effect of wave interference is given in plate 4. It must be borne in mind that in these illustrations advantage is taken of the fact that when once the set of impulses producing a wave has been imparted to a cord, the after effect may be simulated by letting the wave itself move bodily forward, and, similarly, when once the reflected wave is produced, the after effect may be simulated by letting the reflected wave move bodily backward.

Plate 4. Take two strips of thin cardboard, six inches long and one inch wide. and upon these strips draw a set of waves, as represented at figs. 1 and 2 of plate 4; fig. 1 representing the direct, and fig. 2 the reflected waves. Take also another piece of cardboard, about four inches square, and with a sharp penknife cut through the lines *ab*, *bc*, *cd*, *ef*, *fg*, *gh*, fig. 3, *bc* and *fg* being each two inches full, and *ab*, *dc* about a quarter of an inch. Turn the cardboard over, and run the penknife lightly along the dotted lines *ad*, *eh*. The cardboard strips of figs. 1 and 2 are now slipped through the slits *bc*, *fg*, when they appear as in fig. 4. Now push the upper strip from left to right and the lower strip at an equal rate from right to left, and the movements of the cord as influenced by the superposition of the direct and reflected waves are easily seen. In plate 4 the figures are drawn to one-half the scale recommended for the apparatus.

The student may with this apparatus study the interference of a direct and reflected wave of the form represented in fig. 5 of plate 4, when it will be seen that nodes and ventral segments are produced. The subject is further worked out in Reference Chapter C, near end of book, to which the student is referred

Laws of Vibrations of Strings. It will be well to preface the development of the formula for vibration of strings by a few definitions and statements.

1. A string (also a column of air, a rod, or plate) can be set in vibration by a series of impulses applied, and if after a series has been applied the same series be repeated, we have a *periodic* vibration of the string. Usually in the series of impulses there are just as many in an upward or

positive direction as there are in a downward or negative direction. The impulses represented in figs. 1 and 5, plate 3, are examples.

2. If the impulses applied to the string correspond to those of fig. 1, plate 3, we have a simple harmonic motion of each particle of the string, and the curve produced in a single wave is such as represented in fig. 2 of plate 3. If, however, any curve be drawn, such as fig. 5 of plate 3, and impulses be applied corresponding to the abscissæ of this curve, the wave-form produced will correspond to this curve; but, according to the velocity with which the impulses run along the string, the curve may be elongated, as in fig. 5 of plate 4, or compressed, as in fig. 6 of the same plate.

3. When a wave is formed from impulses corresponding to abscissæ of an arbitrary curve, of which fig. 5, plate 3, is an example, the wave may be shown to be made up of a number of waves, each produced by a simple harmonic motion. So far as vibration of strings is concerned in relation to music, we may say that the stationary wave producing a note is made up of a multitude of waves, such that if the length of the simple harmonic wave of the fundamental be λ, the lengths of the other waves present will be $\frac{1}{2}\lambda$, $\frac{1}{3}\lambda$, $\frac{1}{4}\lambda$, ... $1/a\lambda$, where the denominator of the fraction is one of the ordinary numbers (Fourier's Theorem). The number of vibrations n is of course the reciprocal of the wave length. Thus, if with a wave of fundamental length λ there are n vibrations per second, then the pitch of the other vibrations present will be $2n$, $3n$, $4n$ ...... an. Thus, taking the lowest *C* of the pianoforte as of 33 vibrations, we have when this note is sounded (*or may have*) the following notes: *C* of 2 × 33, or 66 vibrations; *G* of 3 × 33, or 99 vibrations; *c* of 4 × 33, or 132 vibrations; and so on indefinitely. This matter will be more particularly referred to in the chapter on the Physical Theory of Music. All that I am anxious here for the student to understand is that a vibration of a string, however complex, may always be resolved into a series of simple harmonic vibrations, of which the number per second must be twice, three times, &c., the number in the lowest or fundamental. These sub-harmonic vibrations are termed *partials.*

"G. S. Ohm was the first to assert that there is only one form of vibration which will give rise to no harmonic upper partial tones, and which will, therefore, consist solely of the prime tone;"* such a form of vibration is the simple harmonic one. "The French mathematician Fourier has proved the correctness of a mathematical law, which in reference to our present subject may be thus enunciated:—*Any given regular periodic form of vibration can always be produced by the addition of simple vibrations. having pitch numbers which are once, twice, thrice, four times, &c., as great as the pitch numbers of the given motion.*" †

4. If a wave-form travel over 1000 centimetres in one second, and the length of each wave is one centimetre there will be 1000 waves produced per second; if the length of each wave were two centimetres, the number formed would be 500; and generally we may say that if v is the velocity of propagation of a wave along a string, n the number of waves formed per second, and λ is the length of each, then

$$v = n\lambda \qquad \text{(i.)}$$

again, since there are n waves formed per second, the time occupied in the formation of each is $1/n$, or if τ is the periodic time of the wave, we have from (i.)

$$\tau = \frac{1}{n} = \frac{\lambda}{v}. \text{ or } \lambda = v\tau \qquad \text{(ii.)}$$

5. When we are dealing with a string of length l, the reflected waves produce nodes and ventral segments, the distance between the nodes being a half-wave length. If the string sound its fundamental, the length of the wave will be double the length of the string; that is, $\lambda = 2l$, and as from (i.) $\lambda = \frac{v}{n}$ we have $\frac{v}{n} = 2l$, or

$$n = \frac{v}{2l} \qquad \text{(iii.)}$$

n here expresses the number of waves per second, but this is obviously the number of vibrations of the string per second, for the string makes one excursion *to* and one *fro*, that is, makes a "swing swang" during the formation of one wave.

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* *Sensations of Tone*, by Helmholtz, 1885, p. 23.

† Ibid. p. 34.

6. Now it may be shown that $v=\sqrt{\frac{t}{m}}$ where t is tension in dynes upon the string, and m is the mass of one centimetre of the string.* Hence, substituting this new value of v in (iii.), we have

$$n=\frac{1}{2l}\sqrt{\frac{t}{m}} \qquad \text{(iv.)}$$

the usual formula for vibration of a string of length l when sounding its fundamental. This formula may be changed about by substituting values of m. Thus, if w is the weight of the string, then $m=w/l$; again, if d is the density of the material of the string, and a its sectional area, then $m=ad$; and, again, if r is the radius of a circular string, since $a=\pi r^2$, we have $m=\pi r^2 d$. Hence, substituting these various values of m, we have

$$n=\frac{1}{2l}\sqrt{\frac{tl}{w}}=\frac{1}{2l}\sqrt{\frac{t}{ad}}=\frac{1}{2l}\sqrt{\frac{t}{\pi r^2 d}}=\frac{1}{2rl}\sqrt{\frac{t}{\pi d}}$$

Maintenance of Nodes and Ventral Segments in Cords. Give to the stretched cord of fig. 2 an impulse represented by the arrow a_1, the impulse is propagated towards A, and also towards B; consider each of these separately. At B the impulse is reversed, as shown at a_2, and it then runs in this direction to A, where it is again reversed, as shown at a_3, and then travels towards B again, coinciding with the original impulse when it reaches a_1. Now the distance $a_1\,a_2+a_2\,a_1+a_1\,a_3+a_3\,a_1$ is twice

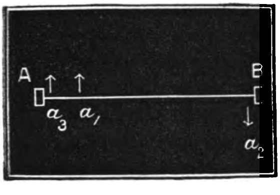

Fig. 2.

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* In a solid generally $v=\sqrt{\frac{e}{d}}$ when v is velocity of the sound wave, e elasticity, or Young's modulus, and d density; but for the velocity of a lateral pulse such as we have been considering in this lesson, we must substitute for e the tension which is acting on unit area of the string; or if t is the actual tension, and a area of the string, we have tension per unit of area $=t/a$; hence $v=\sqrt{\frac{t}{ad}}$, which is the same thing as $\sqrt{\frac{t}{m}}$, where m is the mass of a unit length of the string.

the length of the string. Hence, if the impulse a_1 be repeated at equal intervals corresponding to the time the pulse takes to travel over twice the length of the string, the pulses will be maintained; and if a succession of pulses corresponding to those of either fig. 1 or 5 of plate 3 be applied in regular sequence, each reflected pulse will coincide with each direct pulse after it has travelled over twice the length of the string. If we consider the original pulse travelling towards *A* instead of *B*, the student will easily see that in this case as in the other the pulses will coincide after the pulse has travelled over twice the length of the string.

A violin bow applied at a_1 does in fact impress a series of pulses which set out towards the nut and towards the bridge, and these impulses after travelling twice the length of the string coincide with the original impulse.

EXAMPLE A. With what velocity will a lateral pulse be propagated along a string stretched by a weight of 10 kilogrammes at a place where $g=981$; the mass of one centimetre length of the string being 0·01 gm.?

Here $t=10{,}000\times 981$, and $m=0{\cdot}01$; hence

$$v=\sqrt{\frac{10{,}000\times 981}{0{\cdot}01}}=1000\sqrt{981}=31{,}321 \text{ cm. nearly.}$$

EXAMPLE B. What number of vibrations per second will one metre length of the above string give when sounding its fundamental?

One vibration is produced when the pulse travels over twice the length of the string, that is, the time of one vibration is 200/31321 sec., or the number of vibrations per second is 31321/200 = 156·605.

EXAMPLE C. What is the radius of the string of Example A, knowing that the density of the material of the string is 0·9?

The mass of one cm. length of the string is 0·01 gm.; hence

$$0{\cdot}01=\pi\times r^2\times 0{\cdot}9\times 1, \text{ or}$$

$$r^2=\frac{0{\cdot}01}{\pi\times 0{\cdot}9}, \text{ and}$$

$$r=\sqrt{\frac{1}{90\pi}}$$

$$=0{\cdot}059 \text{ cm.}$$

EXAMPLE D. What is the value of gravity at a place where a string 10 centimetres long, having a mass of 0·01 gm., and stretched by a weight of 100 grammes, vibrates 156 times per second?

$$n=\frac{1}{2l}\sqrt{\frac{t}{m}}$$

Now we know all values but t, and we find t from the expression $\sqrt{t}=2nl\sqrt{m}$, which gives $\sqrt{t}=313{\cdot}2$, and $t=98{,}094{\cdot}24$.

But t is the product of the mass attached to the string (100 grammes) into the value of gravity g; hence

$$g\times 100=98{,}094{\cdot}24$$

$$\text{and}\qquad g=980{\cdot}9$$

EXAMPLE E. A string four feet long is stretched with a weight of 100 lb. at a place where gravity is 32. The string has a mass of ½ oz. per foot. Required the number of vibrations the string will make when sounding its fundamental.

The number n, as shown at page 15, $=\frac{1}{2l}\sqrt{\frac{t}{m}}$, and in example $l=4$ ft., $t=100\times 32$ poundals, $m=1/32$nd lb.; so that

$$n=\frac{1}{2\times 4}\sqrt{\frac{100\times 32}{1/32}}=\frac{10\times 32}{8}=40$$

EXAMPLE F. A string gives a fundamental tone of 128 vibrations per second. What number of vibrations per second will be found in its 5th partial?

By what precedes, the number will be 5 × 128, or 640 vibrations per second.

EXAMPLE G. A string 60 centimetres long when stretched by a weight of 120 grammes ($g=981$) vibrates in four segments, the mass of the string is 0·01 gm. per centimetre. What number of vibrations per second has each of the four segments?

The length of one segment is 60/4, or 15 centimetres, and number of vibrations will be

$$\frac{1}{2l}\sqrt{\frac{w\times g}{m}}=\frac{1}{2\times 15}\sqrt{\frac{120\times 981}{0{\cdot}01}}=114{\cdot}36$$

Longitudinal Vibrations in Rods and Pipes.

Let AB, fig. 3, represent a rod, free at both ends. Give an impulse at a_1. As indicated by the arrow, this impulse travels to B, the end B extends a little, as indicated by the dotted portion. This gives rise to a return pulse, in which the parts are moving towards B, while the pulse itself travels towards A. This

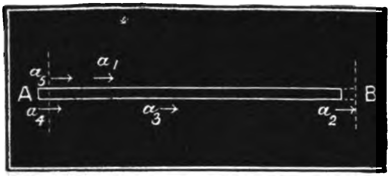

Fig. 3.

is indicated by the arrows a_2, a_3, a_4. At *A* the end of the rod is pulled in, as indicated by the transverse dotted line, and the pulse now travels again towards *B*, with the particles moving also in this direction, as indicated at a_5. Thus the pulse, after travelling *twice* over the length of the rod, coincides with the original impulse.

If at a_1 a series of impulses be given, as indicated in fig. 4 by the arrows 0, 1, 2, 3, ... 0, $\overline{1}$, $\overline{2}$, $\overline{3}$, ... produced, for example, by rubbing to and fro this part of the rod, then the end *B* of the rod vibrates to and fro, like one prong of a vibrating tuning fork as the impulses reach the end *B* in succession; and it is not difficult to see that if the periodic time of the set of impulses coincides with the time required for each pulse to travel over twice the length of the bar, the vibration of the rod will be steadily maintained, and the end *B* will move to and fro once while the complete wave runs twice over the length of the rod. The same remark applies, of course, to the other end *A* of the rod.

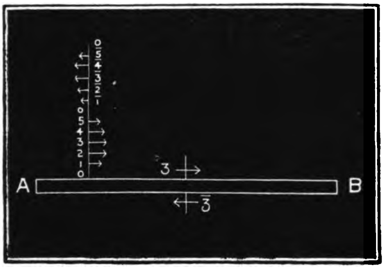

Fig. 4.

The student should think out the effect of transmitting a series of pulses 0, 1, 2, 3, ... 0, $\overline{1}$, $\overline{2}$, $\overline{3}$, ... as shown in fig. 4; they may for the sake of simplicity be propagated at the extremity *A* of the rod. It will be seen, if each pulse be traced in its forward and backward course, that by the time $1 \rightarrow$ reaches the end *B*, $\overline{1} \leftarrow$ will be just starting from *A*; these will then meet in the middle of the rod, thus $\begin{matrix} 1 \rightarrow \\ \overline{1} \leftarrow \end{matrix}$, and simply neutralise each other in passing. All the other impulses similarly meet in the centre of the rod in opposition; hence a node is formed there, while at each end of the rod the direct and reflected pulses coincide.

If the periodic time of the pulses be now halved, the above interferences occur on each half of the rod; hence there will be a node at the centre of each half of the rod, and a ventral segment between. If the periodic time be made one-third, the interferences will take place on each third of the rod, giving rise to three nodes at each half-third, and so on with other periods.

If instead of a rod we are dealing with a tube of air, as in an open organ pipe, precisely the same reasoning applies, and a node is formed in the centre of the tube when the fundamental is sounded.

Kundt's Experiments. *AB*, fig. 5, represents a rod of metal clamped at the centre, while *PQ* represents a glass tube containing air, and closed at one end by a moveable piston. On rubbing *AB* the end *B* moves backwards and forwards, and sets the air in *PQ* in motion. Stationary waves are formed in *PQ*, and the length of the ventral segments is found by strewing lycopodium powder in the tube. Now, double the length of the ventral segments is the length of the wave produced in the air, while double the length of the rod is the length of the wave in the metal. Thus the length *AB* is the length of the half-wave in the metal, while either of the lengths *Pa*, *ab*, *bc*, or *cQ*, is the length of the half-wave in air. Now *AB* in the drawing is three times the length of *ab*, which implies that the velocity of the sound pulse in the metal is three times that in air.

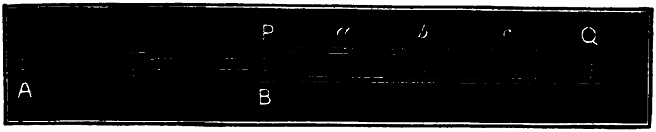

Fig. 5.

EXAMPLE H. A rod of brass, one metre long, used in one of Kundt's experiments, produced stationary waves of air with ventral segments 9·2 cm. long. The temperature of the air at the time was 10°C. Required velocity of sound in the brass.

The velocity of sound in the air (Book C, p. 8) is $33{,}240 + 60 \times 10$, or 33,840 cm. per second, and

$$\frac{\text{Velocity of sound in brass}}{\text{Velocity of sound in air}} = \frac{\text{Length of rod}}{\text{Length of ventral segment}} = \frac{100}{9{\cdot}2} = 10{\cdot}87$$

Hence the velocity in the brass being 10·87 times that in air, and that in air being 33,840 cm, we have for the velocity of sound in brass 33,840 × 10·87, or 367,840·8 cm.

Remarks. Nodes are commonly referred to as places of rest. It must be remembered, however, as already understood from previous statements, they are places where equal and opposite strains are constantly in action; hence alternately plus and minus pressures occur at a node, and but little motion. At a ventral segment, on the contrary, there is motion of the particles, but no pressure.

From node to node is half a wave length; from node to middle of ventral segment is a quarter of a wave length; and as the free end or ends of rods and pipes are always ventral segments, we may easily determine position of nodes when the pitch of a note is given.

EXAMPLE K. An open organ pipe two feet long, and therefore whose fundamental tone is about *c′* (256 vibrations), sounds its fifth overtone. (*a*) What is the pitch of this overtone? (*b*) At what distance from the mouth is a node formed? (*c*) How many nodes are formed altogether?

The overtones of an open organ pipe are as the sequence of numbers 1, 2, 3, 4, 5, ... Hence the fifth overtone will have five times the number of vibrations of the fundamental, that is, 256 × 5, or 1280 vibrations, which is its pitch, or in physical notation *e‴*. The length of the wave will be one-fifth of that of the fundamental, or one-fifth of 4 feet, or 9·6 inches, and the quarter wave is one-fourth of this, or 2·4 inches. Hence a node will be formed at 2·4 inches from each end of this pipe. The total number of ventral segments will be found by dividing the remaining portion of the pipe (24 − 4·8), or 19·2 inches by 4·8 inches, which gives four, showing that there are four ventral segments, and of course five nodes. See fig. 6.

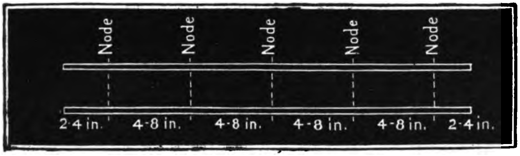

Fig. 6.

Hence it has been found (*a*) 1280, or *e‴*, (*b*) 2·4 inches, (*c*) 5 nodes.

EXAMPLE L. Find pitch, number of nodes, with distance from mouth of pipe of the first node, in a closed organ pipe two feet long, which is sounding its third overtone.

Length of wave of fundamental, 8 feet, or 96 inches. Pitch of third overtone is five times that of fundamental;* its wave length is 96/5, or 19·2 inches. First node will be at one-fourth of this distance from the mouth, that is, 4·8 inches.

Remaining portion of pipe will be ventral segments of half a wave (9·6 inches) each. There will be then 24 in. *minus* 4·8 inches. or 19·2 inches divided into two ventral segments, and there will be three nodes, including the bottom of the pipe. See fig. 7.

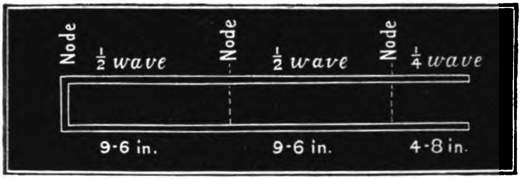

Fig. 7.

Lateral Vibrations of Rods. *When fixed at both ends* and vibrated transversely the divisions are as with strings, 1, 2, 3 ...... *n*; but the number of vibrations of each segment are as the squares of twice the number of segments plus one. Thus, if a rod sounding its fundamental gives a note of 256 vibrations, it will, when divided in four segments, not give 256 × 4, but the two vibrations will be as 9 : 81, so that the number of vibrations will be 256 × 9. Fig. 8 shows these divisions for the first four subdivisions.

When fixed at only one end, if rods are rectangular, the number of vibrations are inversely as square of the length, and directly as thickness.

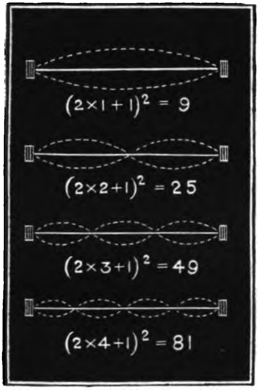

Fig. 8.

$$\frac{\text{Vibrations rod } A}{\text{Vibrations rod } B} = \frac{\text{Thickness of } A}{(\text{Length of } A)^2} \div \frac{\text{Thickness of } B}{(\text{Length of B})^2}$$

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* See Book C, p. 16.

By thickness is meant the thickness of the rod measured in the plane of vibration; as, for example, between the jaws of the vice which holds the rod.

When free at both ends, as in the glass and other harmonica. The simplest case is where there are two nodes, fig. 9 (upper part); the next simplest where there are three. Considering only the ventral segments between the nodes, as between *a*, *b* and *b*, *c* of fig. 9, the relative number of vibrations will be found by doubling the number of ventral segments, adding one, and then squaring, just as in the case of rods fixed at both ends. See page 21.

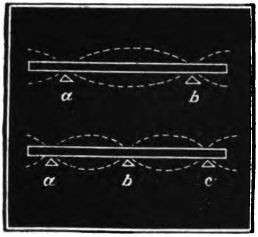

Fig. 9.

The student will have noticed that as regards partials or overtones they are, in the case of flexible strings and organ pipes, in complete harmony with the fundamental, at any rate for the first few partials. Thus, a string giving the note *c′* of 256 vibrations has for its first three partials 2 × 256 or *c″*, 3 × 256 or *g″*, 4 × 256 or *c‴*. But a rod with its ends fixed which gives a fundamental tone *c′* would give notes (25/9) × 256, or *a″*♭ nearly, (49/9) × 256, or *d‴*, and these partials are not at all harmonic with the fundamental.* Now pianoforte strings may be regarded as thin rods, and the thoughtful student may well ask if the upper partials of a pianoforte wire are like strings or like rods. Well, in the first place the rods referred to are *fixed* at the ends, whereas the pianoforte wires pass over bridges. The wires, too, are under great stress, and are thin, and under these circumstances their vibrations resemble those of flexible strings, but require what is called a *correction for rigidity.*

If N_1 are number of vibrations due to a flexible string of length l and radius r, and stretched with T dynes tension per unit of area, and n_1 the number due to a thin string, such as a pianoforte wire, we have (*Donkin's Acoustics*, page 174)

$$n_1 = N_1\left\{1 + i^2 \times \frac{\pi^2}{8} \times \frac{r^2}{l^2} \times \frac{q}{T}\right\}$$

---

* The names of these inharmonic partials are from Helmholtz *Sensations of Tone*, 1885, page 70.

where i is number of the partial, $\pi =$ circumference of a circle divided by its radius 22/7ths about, $q =$ Young's modulus. Now the partials diminish in intensity as they rise higher in the series, so i in the above formula need not be taken as more than a few digits, $\pi^2/8$ is 1·2 about, r^2/l^2 must be a very small fraction, q/T is large, but the effect of all quantities together is to raise a partial but a few vibrations above those of a flexible string.

It may be mentioned here that with the strings giving low notes, that, in order to avoid rigidity, and yet have strings of high density, it is usual to wrap a thin wire or cord round with copper or silver wire. The low notes of the pianoforte and the lowest notes of the violin are produced by this device.

EXERCISE II.

1. A string 100 centimetres long, and stretched with a weight of 981 grammes at a place where $g = 981$, has a mass of one gramme. Find (i.) velocity of the lateral pulse in the string; (ii.) the number of vibrations the string will make per second when sounding its fundamental.

2. The stretching weight is now made one-fourth of 981 grammes. Find the velocity of the lateral pulse. Find also what number of complete vibrations per second must be given to the string per second that it may divide into two ventral segments.

3. A string is stretched by a weight of 100 grammes at a place where $g = 981$, and gives twenty vibrations per second. If the string and weight are removed to a place where $g = 1962$, what number of vibrations will the string give?

If, however, the tension 100 × 981, or 98,100 dynes, were produced by a steel spring instead of a weight, and the arrangement were transferred to a place where $g = 1962$, what would now be the number of vibrations per second? In both examples consider the tension as not affected by the weight of the string.

4. The fourth partial of a string is making 1024 vibrations per second. What length of a string which has a mass of one-hundredth of a gramme per centimetre, and stretched with a weight of 981 grammes, will give this note as its fourth partial?

5. A rod of metal 100 cm. long, when clamped at its centre and rubbed longitudinally, gives the note c''' (1024 vibrations per second). Find (i.) the time the longitudinal pulse takes to travel over twice the length of the rod, or to form one wave; (ii.) the number of centimetres the pulse would travel in the rod in one second; that is, find the velocity of sound in the metal.

6. The rod of Question 5 is connected, as in Kundt's experiment, with a tube containing air at 0°C. What will be length of a ventral segment of stationary waves of air produced in the tube?

7. A rod of copper one metre long, and in which the velocity of sound at 20°C. is 356,000 centimetres per second, is used in Kundt's experiment. The glass tube is filled first with air at 20°C., second with carbon dioxide at 20°C., third with hydrogen at 20°C., when the average lengths of segments were in air 9·7 centimetres, in carbon dioxide 7·6 centimetres, in hydrogen 37 centimetres. From these data calculate velocity of sound in (i.) air, in (ii.) carbon dioxide, and (iii.) in hydrogen at 20°C.

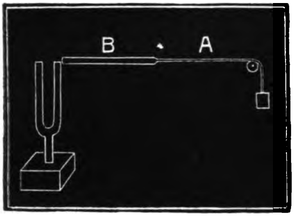

Fig. 10.

8. Two strings, *A* and *B*, see fig. 10, are fastened together. *B* is attached to one prong of a fork giving 256 vibrations per second, while *A* passes over a pulley, and is fastened to a weight of 144 grammes. The length of *A* (measured from the pulley) is equal to that of *B*, while the weight of *B* is four times that of *A*. If on sounding the fork the string *A* forms two ventral segments, what number will be formed on *B*?

9. A rod fixed at both ends when struck transversely gives as fundamental a note of 315 vibrations. What number of vibrations will be given (i.) by the second, (ii.) by the third, and (iii.) by the fifth overtones?

10. A plate of brass with one end free and the other fixed in a vice gives a note c'. To what extent must the plate be diminished in length to give (i.) the note c'''? To what extent (ii.) to give the note c''?

11. A pianoforte wire has a length of 200 centimetres, a radius of 0·06 centimetre, and is stretched by a weight of 100 kilogrammes. The fundamental note of the string is g', 776 vibrations. Find the correction for rigidity for its fifth partial, and compare number of vibrations when thus corrected with the number uncorrected.

ANSWERS.

1. $v=9810$ cm. per second; $n=49·05$.
2. 4905 cm. per second; 49·05 per second.
3. (i.) $20\sqrt{2}$; (ii.) the same, viz., 20.
4. 19·16 cm. nearly.
5. 1/1024th second; 204,800 cm. per second.
6. 16·23 cm.
7. (i.) 34,532 cm. per second; (ii.) i × 7·6/9·7; (iii.) i × 37/9·7.
8. 4 segments.
9. (i.) 875; (ii.) 1715; (iii.) 4235.
10. (i.) must be made one-half the length; (ii.) $1/\sqrt{2}$ of the length.
11. Correction = 0·75; 3880·75 : 3880.

LESSON III.

Pitch and its Measurement. Physical Theory of Music.

Measurement of Pitch. For a description of the syren, an instrument formerly used to determine the pitch of a note, reference may be made to text books, in most of which it is described. Probably, however, the student will already have seen a syren in use at his class lectures, or may have used it in the laboratory.

A more exact method of measuring pitch is by means of a *tonometer*, of which the first was devised by Scheibler in 1834.* Other forms of tonometer are applications of the same principle.

As explained in Reference Chapter B, when two waves travelling with equal velocity but having different wave lengths encounter each other, a composite wave is formed, the amplitude of which rises and falls periodically, giving rise to the phenomenon of *beats*. A few trials on paper with waves of different frequency will soon show the student that this periodic rising and falling is measured by the numerical difference between the wave frequencies. Thus, if two tuning forks, A and B, are sounded together, if A give 256 vibrations per second, and B 260 vibrations per second, then four beats per second will be counted. If A gives 256 vibrations, and B 512 vibrations, there will be $512 - 256$, or 256 beats per second, in this case too many to be discerned by the ear. Generally if N_1 be number of vibrations of fork A, and N_2 be number of vibrations of fork B, the number of beats will be $N_2 - N_1$; or if we express the number of beats by N_b, we may say

$$N_b = N_2 - N_1$$

an equation from which, knowing two of the terms, we can find the third.

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* *See Elementary Lessons on Sound*, by W. H. Stone. Macmillan, 1879, page 100, *et seq*.

Suppose, then, we wish to determine the pitch number of a note on the pianoforte, we select from a series of forks of known value one near the note, and which gives beats with the fork. If the number of beats produced is, for example, four, and the pitch of the fork is 256, we know that the pitch of the note we are trying is either 260 or 252 (the ear will tell which is the higher in pitch—the fork or the note), and so its absolute value is known.

A difficulty is sometimes experienced in judging whether the trial fork is of higher or lower pitch than the note; it may, however, be readily ascertained by loading the fork with a little wax, which "slows" the vibrations of the fork. Then if the beats are fewer per second, this means that the fork was higher in pitch than the note on the pianoforte, while if the beats are more numerous, it means that the fork was lower in pitch.

Scheibler's tonometer is nothing but a series of tuning forks, each of known pitch, and the tonometer is used in the manner just indicated.

The absolute value of the forks in Scheibler's tonometer is determined in the following ingenious manner:—Two tuning forks of unknown pitch are selected as nearly as possible an octave from each other, and then they are finally adjusted by weighting or grinding until they are just an octave apart. This is determined either by the ear, or better by the optical method of Lissajous. If the vibrations due to one fork be N_1, then the other will be $2N_1$. Now take a number of forks n_1, n_2, n_3, ... &c., and determine the number of beats between N_1 and n_1, between n_1 and n_2, between n_2 and n_3, and so on. Suppose the number of beats between N_1 and n_1 are 4, between n_1 and n_2 are 5, between n_2 and n_3 are 4, ... and finally that between the last of the intermediate forks and the octave there are four beats. Now count up the various beats given by successive forks, and add them together. Suppose these make 260, then this shows that the fork N_1 has 260 vibrations, and, of course, the octave has 520 Again, all the vibration numbers of intermediate forks are known, for since the beats between N_1 and n_1 are 4, and N_1 has 260 vibrations, the fork

n_1 has 260 + 4, or 264, and so on with the other forks; and now we have a set of forks by which any note lying within the octave can be determined.

EXAMPLE A. Two tuning forks, known to have an interval of a fifth, but the absolute pitch unknown, gave with ten intermediate forks beats which when added together made up 100. Required the absolute pitch of the two forks.

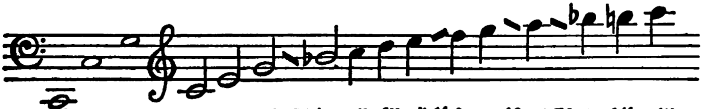
From the formula developed, page 25,

$$N_b = N_2 - N_1$$

but N_b by experiment is 100, and N_2 is ${}^3/_2$ of N_1 since they are a fifth apart; hence

$$100 = {}^3/_2 N_1 - N_1, \text{ or } \tfrac{1}{2} N_1 = 100,$$
$$\text{and } N_1 = 200, \text{ and } N_2 = 300$$

Harmonics—Upper Partials. Sound the *C* of the great octave on the piano, that is, the *C* of 66 vibrations, and listen attentively. Besides the fundamental tone there may be heard the octave, the fifth above the octave, and even other tones may be heard by a skilled and attentive ear. With the help of proper resonators these *partials*, or *harmonics*, or *overtones*, may be easily recognised.* These partials result from the string dividing into 2, 3, 4, ... *n* segments at the same time that the fundamental segment 1 is sounding. The partials that are produced when the note *C* is sounded is given in the following tabular statement, which is copied from Helmholtz *Sensations of Tone*, page 22:—



	C	*c*	*g*	*c′*	*e′*	*g′*	⁷*b′*♭	*c″*	*d″*	*e″*	¹¹*f″*	*g″*	¹³*a″*	⁷*b″*♭	*b″*	*c‴*
No. of Partial	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Pitch Number	66	132	198	264	330	396	462	528	594	660	726	792	858	924	990	1056

The mark placed against the seventh, thirteenth, and fourteenth partials implies that the note is flattened, but that it cannot be expressed exactly in musical notation; while the mark against the eleventh partial signifies a sharpening of the note *f″*.

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* Even a tea cup will bring out some of these partials, the cup being held a short distance from the ear, with its mouth towards the ear.

Difference Tones, Summation Tones. Let two notes, such as c' of 256 and g' of 384, be sounded together; there will be heard a note lower than either of them, consisting of 128 vibrations, that is, the note c. This is a *difference tone*, its vibrations being the difference between the vibration frequencies of the two notes. Again, with the same two notes sounded together, a note can be heard higher than either of them, and consisting of 640 vibrations. This is a *summation tone*, its vibrations being the sum of the vibration frequencies of the two notes.

Intervals. When two notes are sounded together, the vibration frequency of that of higher pitch divided by the vibration frequency of that of lower pitch is called the *interval* between the two notes. Thus the note c' has 256 vibrations per second, the note e' has 320 vibrations, the note c'' has 512 vibrations.

Interval between c' and $e' = 320/256$, or 5/4ths.
,, ,, e' and $c'' = 512/320$, or 8/5ths.
,, ,, c' and $c'' = 512/256$, or 2.
,, ,, c' and $c' = 256/256$, or 1.

Music consists in sounding in rhythm a succession of notes which may have various intervals. Trying, by means of a tonometer, the vibration frequency of the notes met with in modern melodious music in course of an octave, they are as given in the table on page 29,[*] considering a' of 480 vibrations as the key note.[†]

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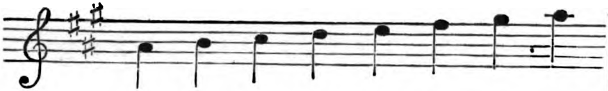
[*] The intervals given in table are those now recognised. Smaller and various other intervals were used by the Greeks. See Blaserna, *Theory of Sound* [H. S. King & Co.], 1876, page 120.

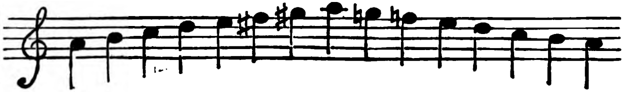
[†] This number is taken as it admits of division easily. It is nearly the number as measured by Degenhardt of the a' of the organ of St. Catherine's in Hamburg. It will be seen by reference to Helmholtz *Sensations of Tone*, translated by Ellis [Longmans, 1885], page 495, that the pitch of the note a' has varied from about 373 (Paris, 1648) to over 500. The concert standard at Albert Hall and Alexandra Palace has $a' = 453·9$ vibrations per second. "The pitch of the Paris Conservatoire is fixed by the rule that *A* (the middle *A* of the piano, or the *A* string of a violin) is to have 435 complete vibrations per second in the tempered gamut." Deschanel *Natural Philosophy*, 1882, part IV., page 900.



NOTES REFERRED TO BELOW.

MAJOR SCALE, KEY A.

	Note.	Absolute Pitch.	Interval from Tonic or Key Note.	Interval from Note to Note.
1	a′	480	a′/a′ = 1	a′/a′ = 1
2	b′	540	b′/a′ = 9/8	b′/a′ = 9/8
3	♯c″	600	♯c″/a′ = 5/4	♯c″/b′ = 10/9
4	d″	640	d″/a′ = 4/3	d″/c♯ = 16/15
5	e″	720	e″/a′ = 3/2	e″/d″ = 9/8
6	♯f″	800	♯f″/a′ = 5/3	f″/e″ = 10/9
7	♯g″	900	♯g″/a′ = 15/8	♯g″/♯f = 9/8
	a″	960	a″/a′ = 2	a″/♯g = 16/15



NOTES REFERRED TO BELOW.

MINOR SCALE, KEY A.

	Note.	Absolute Pitch.	Interval from Tonic or Key Note.	Interval from Note to Note.
1	a′	480	a′/a′ = 1	a′/a′ = 1
2	b′	540	b′/a′ = 9/8	b′/a′ = 9/8
3	c″	576	c″/a′ = 6/5	c″/b′ = 16/15
4	d″	640	d″/a′ = 4/3	d″/c″ = 10/9
5	e″	720	e″/a′ = 3/2	e″/d″ = 9/8
6	f″ *	768	f″/a′ = 8/5	f″/e″ = 16/15
	♯f″	800	♯f″/a′ = 5/3	♯f″/f″ = 25/24
7	g″ *	864	g″/a′ = 9/5	g″/♯f″ = 27/25
	♯g″	900	♯g″/a′ = 15/8	♯g″/g″ = 25/24
	a″	960	a″/a′ = 2	a″/♯g″ = 16/15

* Used in descending.

The various intervals met with, or which are possible in music, have received names. With a key note of *A* the following are some of the more important :—

Unison = the ratio of any note to itself, as a/a, $b/b = 1$, &c.
Major Tone = b'/a', e''/d'', $\sharp g/\sharp f$ - - - - = $\frac{9}{8}$
Minor Tone = c''/b', f''/e'', d''/c'' - - - - = $\frac{10}{9}$
Major Semitone = $d''/\sharp c''$, $a''/\sharp g''$, c''/b', f''/e'' - = $\frac{16}{15}$
Minor Semitone = $\sharp f''/f''$, $\sharp g''/g''$ - - - - = $\frac{25}{24}$
Minor Second = $g''/\sharp f''$ - - - - - - = $\frac{27}{25}$
Major Third = $\sharp c''/a'$ - - - - - - = $\frac{5}{4}$
Minor Third = c''/a' - - - - - - = $\frac{6}{5}$
Perfect Fourth = d''/a' - - - - - - = $\frac{4}{3}$
Perfect Fifth = e''/a' - - - - - - = $\frac{3}{2}$
Major Sixth = $\sharp f''/a'$ - - - - - - = $\frac{5}{3}$
Minor Sixth = f''/a' - - - - - - = $\frac{8}{5}$
Major Seventh = $\sharp g''/a'$ - - - - - - = $\frac{15}{8}$
Minor Seventh = g''/a' - - - - - - = $\frac{9}{5}$

By help of the table the student can easily find the interval between any two notes, bearing in mind that the interval is the ratio of the higher number to the lower. This must not be confused with the summation of intervals, which means their multiplication. Thus the interval between c and $\sharp c$ is 5/4 divided by 6/5, or 25/24; between g and $\sharp f$ it is $9/5 \div 5/3$, or 27/25, &c. The interval between a' and $\sharp c''$ is the *sum* of the intervals 9/8 and 10/9, or $9/8 \times 10/9 = 5/4$, which is just the same thing as $\sharp c''/a$.

The student should notice that the interval between the fundamental and the major third is 5/4, or 25/20, while the interval between the fundamental and the minor third is 6/5, or 24/20; or the interval between the fundamental and the major third is 25/24 greater than that of the minor scale; hence the names major and minor.

Temperament. It will be obvious on looking at the table on page 29, that with 11 strings properly tuned to give 480, 540, &c., vibrations per second, we could sound any notes of the major or minor scale for one octave, as far as the key of *A* is concerned. But suppose we wish to form a major and minor scale starting with b' of 540 vibrations, we shall want several more strings, for we must have for the major scale strings which will give us 9/8 × 540, or 607·5; 5/4 × 540, or 675; 4/3 × 540, or 720; 3/2 × 540, or 810; 5/3 × 540, or 900; 15/8 × 540, or 1012·5; and 2 × 540, or 1080. And for the minor scale strings, which will give us in addition vibrations of 6/5 × 540, or 648; 8/5 × 540, or 864; and 9/5 × 540, or 972; that is, the notes will have vibrations:—

Major Scale.		Minor Scale.	
b'	540	*b'*	540
c''♯	607·5	*c''*♯	607·5
d''♯	675	*d''*	648
e''	720	*e''*	720
f''♯	810	*f*♯	810
g''♯	900	*g''*♯	900
		g	864
a''♯	1012·5	*a''*♯	1012·5
		a''	972
b''	1080	*b''*	1080

Neglecting then the note *b''* of 1080 vibrations, there will be required, in order to run up and down the gamut in both scales, commencing with *b'*, no less than six additional strings, viz.:—For *c''*♯ of 607·5, *d''*♯ of 675, *f''*♯ of 810, *a''*♯ of 1012·5, *d''* of 648, and *a''* of 972 vibrations. Commencing the gamut with *c''*, *d''*, *e''*, &c., new strings must be introduced until we shall find that to play the scales, major and minor, in proper tune, commencing in succession on each of the ten notes of the original scale, and omitting the octave, no less than thirty-nine strings will be required.

Take a piece of paper, and rule eleven horizontal and eleven vertical lines, thus dividing the paper into one hundred squares. Put in the left-hand column, commencing at the top, the successive intervals from the tonic, viz., 1, 9/8, 6/5, 5/4, 4/3, 3/2, 8/5, 5/3, 9/5, 15/8. This column will represent then the relative vibrations required to give the complete scale (major and minor) if the note 1 be taken as tonic. To find the relative vibrations required when the second note of 9/8 vibrations is taken as tonic, head the second column with the fraction 9/8, the number of vibrations for the next note higher will be 9/8 of 9/8, or 81/64; for the following one it will be 6/5 of 9/8, or 27/20, and so on; head the third column with the fraction 6/5, and then take successively 9/8 of 6/5, 6/5 of 6/5, and so on. We shall then have filled up the hundred squares with a series of fractions representing the vibration frequencies required to play the scales for one octave, commencing with each note in succession; now striking out fractions, which are mere repetitions, we shall have thirty-nine fractions remaining. But suppose we wish to have a set of wires, not only for playing *one* complete octave starting from each note in the first column, but to play scales on each of the new notes introduced, we shall find many more strings are required; in fact, Mr. A. J. Ellis, *Proceedings of the Royal Society*, vol. xiii., page 98, determines that seventy-two notes in the octave would be required; so that a perfect piano would require not simply twelve keys, seven white and five black in the octave, but seventy-two keys, and for an ordinary seven octave piano we should require 504. It is out of the question to introduce such a number of notes, and so various schemes have been adopted for reducing the number. Thus, with those which are near to each other, an intermediate note is introduced, which will serve for either.

By contrivance of this kind a *tempered* scale is produced, by means of which an approximate major or minor scale may be obtained, no matter what may be the key note. Of modes of tempering, that which is known as equal temperament is in general use. There are twelve notes in the scale, and their relative vibration frequencies are 1, $1 \times 2^{\frac{1}{12}}$, $1 \times 2^{\frac{1}{12}} \times 2^{\frac{1}{12}}$, or $1 \times 2^{\frac{2}{12}}$, $1 \times 2^{\frac{1}{12}} \times 2^{\frac{1}{12}} \times 2^{\frac{1}{12}}$, or $1 \times 2^{\frac{3}{12}}$ ...... $1 \times 2^{\frac{10}{12}}$, $1 \times 2^{\frac{11}{12}}$, $1 \times 2^{\frac{12}{12}}$, or 2. Thus taking *A* as 480 vibrations, the number for *A*♯ would be $480 \times 2^{\frac{1}{12}}$, or 508·54; for *B* $480 \times 2^{\frac{2}{12}}$, or 538·8. Now *B* should properly have 9/8 times the vibrations of *A*, that is, it should have 540 vibrations, so that in the key of *A* major the note *B* would be a little flat. On the other hand, since with the tempered scale *C*♯ will have $480 \times 2^{\frac{4}{12}}$, or $480 \times 2 = 570·8$. If we were playing in *C* major, *B*, which ought to have $15/16 \times 570·8$, or 535·1 vibrations, will be sharp.

EXAMPLE A. The note c' on a pianoforte was found to have 300 vibrations per second. What number of vibrations must g' have (i.) to be a perfect fifth, (ii.) to have the fifth according to the equal temperament scale?

The perfect fifth will, of course, have $3/2 \times 300$, or 450 vibrations.

The equal temperament fifth, since there are seven semitones between the notes, will be

$$300 \times 2^{\frac{7}{12}}$$

Referring to *A B C Logarithms*,* we have Log. $2 = 0{\cdot}30103$, and 7/12ths of this $= 0{\cdot}17560$, the number corresponding to which (see page 9 of *A B C Logarithms*) $= 1{\cdot}498$ approximately, and

$$300 \times 1{\cdot}498 = 449{\cdot}4$$

so that the number in the tempered scale is a little flat.

Simple Ratios. The Pythagorean Scale. The various intervals which have been referred to have all been derived by what may be called a practical method, that is, the intervals are considered as fixed by a general concensus of opinion on the part of musicians as to what are notes of a scale, the ratios being afterwards obtained by trial as to the vibration frequency of the different notes of the scale.

The Pythagorean Scale of seven notes may be said to be a scale constructed from the two simple ratios 2/1 and 3/2. These two ratios give as to begin with the notes

C	*G*	*c*
1	$\frac{3}{2}$	2

If we now take a fifth below *c*, that is, 2/3 of *c*, we have $2 \div 3/2$, or 4/3, we thus get the note *F*. Again, take the fifth of *G*, and we have 9/4, and lower this an octave, which gives us 9/8, or the note *D*. Now take the fifth of *D*, that is, $9/8 \times 3/2$, and we have the note $A = 27/16$. Raise *A* one-fifth, and then lower it an octave, and we have $E = 81/64$. Lastly, raise the note *A* two fifths in succession, and we have $27/16 \times 3/2 \times 3/2$, or 243/64, and lower this an octave, which gives 243/128, the note *B*. We have then from the two simplest ratios of 2/1 and 3/2 obtained the following or Pythagorean Scale:—

C	*D*	*E*	*F*	*G*	*A*	*B*	*c*
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

---

* *A B C Five Figure Logarithms*, by C. J. Woodward. See advertisement at end of book.

It will be noticed that with the exception of the notes *E*, *A*, and *B*, the intervals are the same in the Pythagorean as in the modern diatonic scale. To tune the harp to this scale the musician need only be able to recognise a fifth and octave, and all the notes could be accurately obtained.

Consonant and Dissonant Intervals. When two notes are sounded together, they may or may not produce harmony. When the interval is harmonic or consonant, a pleasing effect is produced; when the interval is inharmonic or dissonant, an unpleasing effect is produced on a musical ear. So far this is a mere statement, and a musical ear simply decides. Helmholtz, however, gives a reason why some notes sounded together should be harmonious, while others are not so. The propositions involved may be thus stated:—

1. When beats occur between two notes, or between the partials of two notes, then if the tones or partials producing the beats are less than a minor third apart; and, further, if the number of beats per second lie between 5 or 6 per second, and some 60 or 70 per second, an unpleasant effect is produced. If the number of beats are about 78, or more than 78, the harshness vanishes. Hence to determine by this view whether two notes are in harmony, determine the number of vibrations of each note, and of several of their upper partials. By subtraction determine the number of beats. Then, if the number of beats be fewer than 78 about, there will be dissonance.

The number of vibrations of two pairs of notes and their partials are given in the table on page 35; one pair is consonant, the other dissonant.

If instead of c', g', we try the notes C_1 of 33, and G_1 of 49·5 vibrations, it is quite true that beats of less than 78 vibrations will be found, but the tones or partials producing the beats are in all cases more than a minor third apart. Problem 11 of Exercise III. is intended to bring this point out more clearly.

TWO NOTES WELL KNOWN TO BE CONSONANT, *c'*, *g'*.

Note.	Partial.	Vibration No. v_1	Note.	Partial.	Vibration No. v_2	No. Beats. $v_2 - v_1$
c'	1	256	*g'*	1	384	128
c''	2	512	*g''*	2	768	256
g''	3	768	*d''*	3	1152	384
c'''	4	1024	*g'''*	4	1536	512
e'''	5	1280	*b'''*	5	1920	640
g'''	6	1536	*d'''*	6	2304	768

TWO NOTES WELL KNOWN TO BE DISSONANT, *b*, *c'*.

Note.	Partial.	Vibration No. v_1	Note.	Partial.	Vibration No. v_2	No. Beats. $v_2 - v_1$
b	1	240	*c'*	1	256	16
b'	2	480	*c''*	2	512	42
f'	3	720	*g''*	3	768	48
b''	4	960	*c'''*	4	1024	64
d''	5	1220	*e'''*	5	1280	60
f''	6	1440	*g'''*	6	1536	96

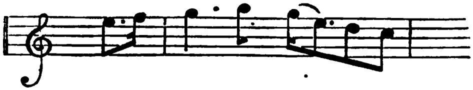
It is easily seen that the partials of the notes *c'*, *g'*, give in no case beats under 128 per second. While with the notes *b* and *c'* a semitone apart, it is not until we get to the fifth partial that the number of beats reaches above 78. Hence here is a physical reason for the dissonance in the second case, and consonance in the first.

EXERCISE III.

1. Two tuning forks, giving 450 and 454 vibrations per second, are sounded together. How many beats per second will be produced? What is the interval between the forks?

2. A fork *P* gives 250 vibrations per second, and a fork *Q* 240 vibrations per second. How many beats per second will be produced? What is the interval between the two forks (i.) expressed as a ratio, (ii.) as a name?

3. Assuming the key note *C* of the following to have 512 vibrations per second, find intervals from note to note, and also the interval of each note as compared with the tonic.



4. In order to determine the pitch of a note on a piano, the note was sounded with a fork *P*, a little lower in pitch, and giving 260 vibrations per second, when four beats per second were heard. What was the pitch of the note?

5. Nine forks, *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *K*, were taken, and it was found that with *A* and *B* there were three beats per second, with *B* and *C* four beats per second, with *C* and *D* two beats per second, with *D* and *E* five beats per second, with *E* and *F* five beats per second, with *F* and *G* four beats per second, with *G* and *H* five beats per second, and with *H* and *K* five beats per second. *K* was found by an accurate ear to be just one octave above *A*. Find the absolute pitch of each of the forks.

6. A series of tuning forks, *A*, *B*, *C*, *D*, were found to give *A* 480, *B* 960, *C* 720, *D* 600. What are the intervals between *A* and *B*, *B* and *C*, *C* and *D*, *B* and *D*? Write these notes in the treble clef.

7. What must be the vibration numbers of two forks that will divide the octave a', a'', into three equal intervals? What are the names of the nearest notes to those obtained?

Here we have to find a ratio which, multiplied into itself three times, will give 2. The cube root of 2 will satisfy this, for $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2$.

8. When two forks of 240 and 360 vibrations are sounded together, a difference tone is heard much lower than either fork. What is the name and vibration frequency of this difference tone?

9. What are the partials as far as the sixth partial of the note C_1 of 33 vibrations per second?

10. What are the partials as far as the sixth partial of the note G_1 of 49·5 vibrations?

11. State the intervals between the above notes, and also between their successive partials, and give the number of beats produced. When thus analysed are the notes consonant or dissonant?

12. Make out a table similar to that of page 35, having reference to the following pairs of notes c', e', and c', d'.

ANSWERS.

1. 4, 227/225.
2. 10, 25/24, or a minor semitone.
3. $f/e=16/15$, or a major semitone; $g/f=9/8$, major tone; $a/g=10/9$, minor tone; $g/e=6/5$, minor third; $e/d=10/9$, minor tone; d/c 9/8, major tone. Intervals from tonic are $e/c=5/4$, major third; $f/c=4/3$, perfect fourth; $g/c=3/2$, major fifth; $a/c=5/3$, major sixth; $d/c=9/8$, major tone.
4. 264.
5. $A=33$, $B=36$, $C=40$, $D=42$, $E=47$, $F=52$, $G=56$, $H=61$, $K=66$.
6. B/A octave, B/C major third, C/D minor third, B/D minor sixth.
7. a', $\sqrt[3]{2}\,a'$, $\sqrt[3]{2} \times \sqrt[3]{2}\,a'$, or $2^{\frac{2}{3}}\,a$, $2\,a'$, or in numbers approximately as found by logarithms, a', 1·260 a', 1·587 a', 2 a'.
8. $A=120$.
9. $C_1=33$, $C=66$, $G=99$, $c=132$, $e=165$, $g=198$.
10. $G_1=49·5$, $G=99$, $D=148·5$, $g=198$, $b=247·5$, $d=297$.
11. All the required intervals are fifths; the beats are 16·5, 33, 49·5, 66, 82·5, 99. The notes are consonant, as none of these partials are nearer than a fifth.

Note on Equal Temperament. It is shown from the table on page 29 that 11 notes are required for each octave, in order to play melodious music. The 12th note will, of course, be the octave of the first. Hence if m is the number of vibrations of the key note of an octave, $2m$ will be the number for its octave. If we then introduce 10 notes of vibration numbers n/m, o/n, &c., and make all the ratios equal, we shall have the series—

$$\begin{array}{ccccccccccccccccccccccc}
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 & & 10 & & 11 & & 12 \\
\frac{n}{m} & = & \frac{o}{n} & = & \frac{p}{o} & = & \frac{q}{p} & = & \frac{r}{q} & = & \frac{s}{r} & = & \frac{t}{s} & = & \frac{u}{t} & = & \frac{v}{u} & = & \frac{w}{v} & = & \frac{x}{w} & = & \frac{2m}{x}
\end{array}$$

Now if we call R the ratio of any one of the above fractions, we have, multiplying them together, the quotient R^{12}. But also multiplying them together we have $\frac{n}{m} \times \frac{o}{n}$, &c., $=2$, so that

$$R^{12}=2$$

and $R=2^{\frac{1}{12}}$, that is, the common ratio of the notes is $2^{\frac{1}{12}}$, as given at page 32.

LIGHT.

LESSON IV.

Reflection from a Single Plane Mirror, from Mirrors inclined to each other, and from Curved Mirrors.

For geometrical demonstrations in this and following lessons the student is referred to Reference Chapter A, near end of book.

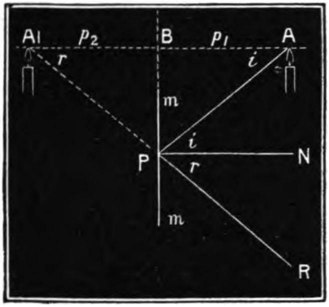


Fig. 11.

Image formed by a Plane Mirror. Let *A*, fig. 11, be a point of light (the extreme tip of a candle flame, for example,) from which rays fall on to the mirror *mm*. Produce *mm*, and draw *ABA'* perpendicular to this produced surface. Draw *AP* representing *any* ray which falls upon the mirror, and draw the normal *PN*. This ray will be reflected in the direction *PR*, making the angle of reflection *r* equal to the angle of incidence *i*. Produce *RP* backwards until it meets *ABA'* at *A'*. Now *PN* and *ABA'* are parallel lines; hence the angle at *A* is equal to the angle *i*, and also the angle at *A'* is equal to the angle *r*; but $i = r$, that is, the angles at *A* and *A'* are equal to each other. Again, the angles at *B* are equal, being right angles; so that in the triangles *ABP* and *A'BP*, two of the angles are equal, each to each, and they are on the same base *BP*, so that they are equal to each other in every respect; hence $AB = BA'$. Since *AP* was *any* ray, it follows that all rays falling on the mirror will be reflected as though they proceeded from the point *A'*. That is, *All rays proceeding from a luminous point, and falling upon a plane mirror, are reflected from the mirror as though they* *came from a point perpendicularly as far behind the mirror, or the mirror produced, as the luminous point is in front of the mirror, or mirror produced, and both perpendiculars are in the same straight line.*

If instead of one point we have an assemblage of points forming an object, then for every point of the object there will be a corresponding point in the image, the position of each point being determined by the proposition given above, the result being a virtual image corresponding in every respect to the object, excepting that it is laterally inverted.

Angular deviation of a ray incident upon one mirror and thence reflected to a second mirror. A ray PO, fig. 12, falls upon a mirror MN, making an angle of incidence a°. Instead of taking the direction OQ, it now takes the direction OQ^1, being turned or deviated in the positive direction (see Reference Chapter A) through the angle QOQ^1. But $QOQ^1 = 180^\circ - 2a^\circ$. Hence deviation at MN is $180^\circ - 2a^\circ$. Or if D = this first deviation, then

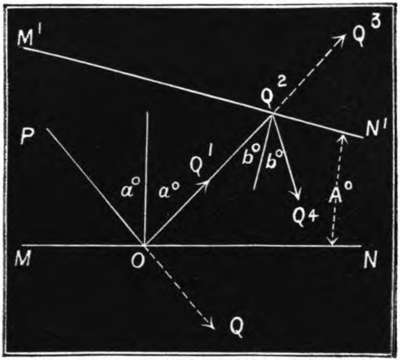


Fig. 12.

$$D = 180^\circ - 2a^\circ$$

Let now the reflected ray OQ^1 fall upon a second mirror M^1N^1 at an angle of b° with the normal, the angle between the mirrors being A°. The ray instead of continuing the direction Q^2Q^3 will be turned in the negative direction through the angle $Q^3Q^2Q^4$, that is, it will be turned through the angle $-180^\circ + 2b^\circ$. Now if the mirror M^1N^1 had been parallel to MN, then the angle b would have equalled the angle a; but every degree of inclination of the mirrors takes off a degree from this angle; so that if A° is the inclination of the mirrors, then the angle b will be $a^\circ - A^\circ$, and we may write $a^\circ - A^\circ$ for b; so that if D_2 equal deviation at second reflection, we have

$$D_2 = -180^\circ + 2b = -180^\circ + 2a^\circ - 2A^\circ$$

If the ray Q^2Q^4 fall again upon *MN*, producing a third reflection, then, as regards this third reflection, if *c* is the angle at which the ray strikes the first mirror *MN*, we have similarly the deviation, as

$$180^\circ - 2c^\circ$$

$$\text{But } c^\circ = b^\circ - A^\circ = (a^\circ - A^\circ) - A^\circ = a^\circ - 2A^\circ$$

Hence for the third deviation we have

$$D_3 = 180^\circ - 2\ (a^\circ - 2A^\circ) = 180^\circ - 2a^\circ + 4A^\circ$$

Following up this reasoning, we can express the deviation at each reflection in terms of *a* and A°, and we obtain the following expressions, using D_1, D_2 ...... D_n for the number of deviations, a° for the angle of incidence, and A° for the angle between the mirrors.

$$\begin{aligned} D_1 &= 180^\circ - 2a^\circ \\ D_2 &= -180^\circ + 2a^\circ - 2A^\circ \\ D_3 &= 180^\circ - 2a^\circ + 4A^\circ \\ D_4 &= -180^\circ + 2a^\circ - 6A^\circ \\ D_n &= \pm 180 \mp 2a^\circ \pm 2\ (n-1)A^\circ \end{aligned}$$

the upper signs being used if *n* is odd, and the lower signs if *n* is even.

If we want to find the total deviation after *n* reflections from mirror to mirror, we must, of course, add up the successive deviations, that is, find the sum of $D_1 + D_2 + D_3$ ...... $+ D_n$, and it will be found

(1) If *n* is odd, $D_1 + D_2 \ldots\ldots + D_n = 180^\circ - 2a^\circ + (n-1)A^\circ$

(2) If *n* is even, ,, ,, $= -nA^\circ$

The student should work out a few examples by construction, taking care, in order that there may be no confusion of signs, that the point of junction of the mirrors is as in fig. 12—to the right hand.

EXAMPLE A. Two mirrors are inclined to each other at an angle of 30°. A ray of light falls upon one of them, making an angle of incidence of 20°. What angle will the incident and reflected rays enclose (i.) after three and (ii.) after four reflections?

By formula developed above

(i.) $$D_1 + D_2 + D_3 = 180^\circ - 2 \times 20^\circ + 2 \times 30^\circ$$
$$= 200^\circ$$

(ii.) $$D_1 + D_2 + D_3 + D_4 = -nA^\circ = -4 \times 30^\circ = -120^\circ$$

EXAMPLE B. Two mirrors are inclined to each other at an angle of 30°; a ray of light makes on one of the mirrors an angle of incidence of 60°, that is, the ray is parallel to the second mirror. Required the number of reflections that will occur before the reflected ray is parallel to the incident ray, but opposite in direction.

The emergent ray and the incident ray make an angle of 180°, that is, the deviation is 180°. Hence

$$180^\circ = -n \times 30 \text{ if } n \text{ is even, and}$$
$$180^\circ = 180 - 2 \times 60 + (n - 1) \times 60 \text{ if } n \text{ is odd.}$$

$$\text{If then } n \text{ is even, } n = \frac{180}{30} = 6$$

$$\text{If } n \text{ is odd } (n - 1) = \frac{-60}{60} = 1 \text{ and } n = 0 \text{ (impossible).}$$

The number of reflections then will be 6.

This result the student should verify by means of a drawing.

The deviation of a ray striking a mirror is, as we have seen, $180^\circ - 2a^\circ$ when a is the angle of incidence. Suppose the mirror turned through a small angle b° so as to diminish the angle of incidence by b°. The angle of incidence is now $a - b$, and the deviation will be $180 - 2(a^\circ - b^\circ)$, or $180 - 2a^\circ + 2b^\circ$, that is, the ray is turned $2b^\circ$ when the mirror is turned b°. This shows, then, that with a rotating mirror the angular movement of the reflected ray is twice that of the mirror.

Reflections from two mirrors inclined to each other. In fig. 13, PP', QQ', are traces of two plane mirrors placed at right angles to the plane of the paper, and making an angle A° with each other. The mirrors themselves are represented by thick lines, and their continuations by dotted lines. With the intersection of the planes of the mirrors as centre, describe the portion of the circle PQ. Let a number of objects be placed between the two mirrors, the space between the mirrors then forms what we will call the pattern sector, or sector o. This sector will be bodily reflected by each of the mirrors P and Q, every object in

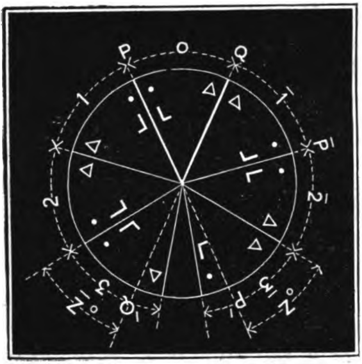


Fig. 13.

the sector, including the portion of the circle PQ,* having its image formed according to the statement on page 38. Thus the sectors 1 and $\bar{1}$ are formed; the sector $\bar{1}$ becomes now an object to the mirror P, and the sector 2 is produced. Similarly sector 1 produces sector $\bar{2}$.

Sector $\bar{2}$ produces sector 3
Sector 2 ,, ,, $\bar{3}$
Sector $\bar{n}$,, ,, $n+1$
Sector n ,, ,, $\overline{n+1}$

and so on, until the sectors, or any portions of them, fall behind the mirrors (in fig. 13 portions of the sectors 3 and $\bar{3}$ are behind). At this point the formation of *complete* sectors ceases. Consider now how to determine the number of complete sectors reflected, leaving until afterwards what further takes place.

Look now at the figure. The sectors have all the same number of degrees as the pattern sector; hence

$$\text{sectors } \bar{1}, \bar{2}, \bar{3} = \text{sectors } 0, \bar{1}, \bar{2}$$

Add to the sectors 0, $\bar{1}$, $\bar{2}$, the part sector Z°, and we have (since PP' is a straight line)

$$0+\bar{1}+\bar{2}+Z^\circ = 180^\circ;\ \text{or}$$
$$\bar{1}+\bar{2}+\bar{3}+Z^\circ = 180^\circ$$

Now if each sector has A°, and n denotes the number of complete sectors at the back of QQ',

$$nA^\circ + Z^\circ = 180^\circ \qquad \text{(i.)}$$

or

$$n = \frac{180^\circ}{A^\circ} - \frac{Z^\circ}{A^\circ}$$

Now as Z° is less than A°, the fraction Z°/A° is less than 1, and as n must be a whole number, n must be the greatest whole number contained in the fraction $180^\circ/A^\circ$, that is,

$$n = \frac{180^\circ}{A^\circ}$$

disregarding any remainder.

Knowing now the value of n, the value of Z° is found by equation (i.), for

$$Z^\circ = 180^\circ - nA^\circ$$

---

* The student should be able to show that PQ will by reflection produce always portions of the same circle.

EXAMPLE C. Two reflectors, PP', QQ', as in fig. 13, are inclined to each other at an angle of 23°. Required the number of complete sectors formed behind QQ', and also magnitude of the part sector Z.

Since 23° will go into 180° seven times, the number of complete sectors will be seven, and the magnitude of the part sector will be $180^\circ - 7 \times 23 = 180^\circ - 161^\circ = 19^\circ$. The part sector then is 19°.

In the same manner the number of sectors and the value of the part sectors behind PP' is determined by the equation

$$nA^\circ + Z^\circ = 180^\circ$$

The effect of the mirror PP' on the part sector Z°, and the effect of the mirror QQ' on the similar part sector $\bar{Z}^\circ$, has not been referred to. PP' will reflect the part sector Z°, QQ' similarly the part sector $\bar{Z}^\circ$. The effect of these reflections is illustrated in fig. 14, and it will be seen that the symmetry of the pattern produced in front of the mirrors is lost in the sector formed behind the mirrors. This point will be referred to again presently.

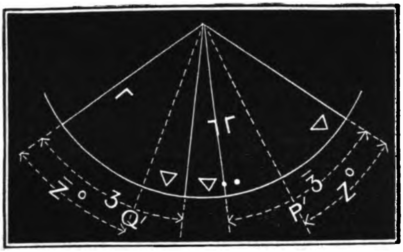


Fig. 14.

Suppose the pattern sector to contain a number of objects, as in fig. 13, the number and character of the reflections can be determined.

The student should first note that the odd sectors $\bar{1}$, 1, $\bar{3}$, 3, &c., have the images laterally inverted, while the even sectors, $\bar{2}$, 2, &c., have the image without such inversion. In considering the last image formed, since it is produced by a reflection of part of the last sector, we can easily determine whether it will be laterally inverted or not.

A further point may be noticed. As the angle A gets less, the part sector Z° becomes greater, and when it equals A° there is no part sector to reflect; the number of sectors formed behind QQ' will be $180^\circ/A^\circ$, and the number behind PP' similarly will be $180^\circ/A^\circ$, or together $360^\circ/A^\circ$; but as the last sector of each series of reflections coincide in the part $Q'P'$, the actual number of reflections will be

$$\frac{360^\circ}{A^\circ} - 1$$

EXAMPLE D. A point is placed three degrees from the mirror Q in fig. 13. The angle between P and Q is 23°. Required how many images of this point will be produced.

From Example C we see that the number of complete reflected sectors behind QQ' are seven, and similarly the number behind PP' are seven; there are, therefore, at least fourteen images of the point, irrespective of the part sectors in front of PP' and QQ'. The part sector (see Example C) is 19°; so that 19° of the pattern sector measured from Q, and 19° of it measured from P, are reflected in addition.

Now 19° measured from P will not contain a point at three degrees from Q, but 19° measured from Q does contain it; so there will be fourteen images due to the complete sectors, and one more due to the part sectors, or fifteen in all.

EXAMPLE E. In the above example, supposing the point still at 3° from Q, but the angle PQ to be 20°, instead of 23°, how many images would be produced?

The number of sectors behind Q will be 180/20, or 9, and the number behind P will also be 9, and there is no part sector. As there is no part sector, the reflections coincide in the sector $P'Q'$; so that the total number of images will not be 18, but 17. NOTE.—In this case it is immaterial the number of degrees the point is from Q, as all points within the pattern sector will appear in the sector $P'Q'$.

EXAMPLE F. Will objects in the pattern sector of Example E appear laterally inverted or not in the sector $P'Q'$?

As the last sector behind PP' is an odd one, viz., the 9th, the image will be laterally inverted. The same is true if the sectors are counted behind PP'; so that in the last sector $P'Q$ the images of object produced by the overlapping of the last reflections coincide in all respects.

The method of determining the number of images of a luminous point placed between them is generally treated in books in a different manner from the one just described. See Goodwin's *Course of Mathematics*, page 511; Parkinson's *Optics*, page 23; Heath's *Geometrical Optics*, page 40; Aldis' *Geometrical Optics*, page 50. The analysis as treated in these works leads to the statement that the number of images formed by the pencil which strikes the mirror P is the whole number next greater than

$$\frac{180^\circ - a^\circ}{A^\circ}$$

and the number formed by the pencil which first strikes the mirror Q is the whole number next greater than

$$\frac{180 - b^\circ}{A^\circ}$$

where a° is the number of degrees the point is from P and b°, the number of degrees the point is from Q, and A° is the angle between the mirrors. Thus, if the angle between the mirrors is 21°, and the luminous point is 1° from P, the number of images formed by the light which strikes P first will be the next whole number to $(180-1)/21$, or 9. While the number of images formed from the light which strikes Q first will be the whole number next greater than $(180-20)/21$, or 8, so that the number of total images will be 17.

Reflection from curved (spherical) surfaces. Let A, fig. 15, be a point of light, rays from which fall on the concave mirror DE. Let AE be one ray which coincides with the principal axis, while AD is *any other* ray which strikes the mirror. The incident ray AD after reflection will fall at B, making the angle r, measured from the radius DC, equal to the angle i, measured from the same radius.

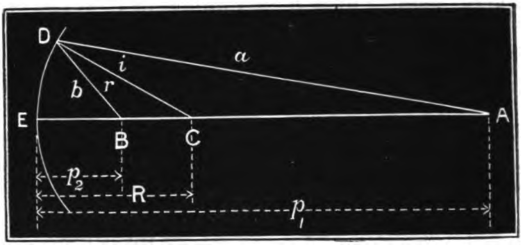


Fig. 15.

Now calling p_1 the length AE, p_2 the length BE, and R the length EC, our first point is to show that

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{2}{R}$$

Call a the distance AD, and b the distance BD, then, as the angles i and r are equal, we have (see Euclid, VI., 3, also Reference Chapter A)

$$\frac{a}{b}=\frac{AC}{BC}=\frac{p_1-R}{R-p_2}$$

But if the mirror is of small aperture $a=AE=p_1$, and $b=BE=p_2$; so writing p_1 for a, and p_2 for b, we have

$$\frac{p_1}{p_2}=\frac{p_1-R}{R-p_2}$$ Multiplying up we have

$$p_1R - p_1p_2 = p_1p_2 - p_2R, \text{ or}$$

$$R(p_1 + p_2) = p_1p_2. \text{ Dividing by } p_1p_2R,$$

we have

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{2}{R} \qquad \text{(i.)}$$

Suppose in formula (i.) that R is very great, that is, that the mirror is of very slight curvature, then the expression $2/R$ is very small; and if R be infinite, as it would be if the mirror were plane, then $2/R = 0$, and we have

$$\frac{1}{p_1} + \frac{1}{p_2} = 0, \text{ or } \frac{1}{p_1} = -\frac{1}{p_2}$$

but this is the case of a plane mirror, as shown at page 38.

Again, in formula (i.) let p_1 be infinite, as would be the case when the rays are parallel, then we have

$$\frac{1}{\infty} + \frac{1}{p_2} = \frac{2}{R}, \text{ or } \frac{1}{p_2} = \frac{2}{R}, \text{ or inverting}$$

$$p_2 = \tfrac{1}{2}R$$

but when p_1 is infinite, or the rays parallel, p_2 is called the length of the principal focus, and this we see is $\frac{1}{2}R$, or one-half the radius of the mirror. Hence as

$$f = \tfrac{1}{2}R, \quad R = 2f, \text{ or } \frac{2}{R} = \frac{1}{f} \qquad \text{hence}$$

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{f} \qquad \text{(ii.)}$$

Let the point of light A, fig. 15, gradually move up towards the mirror; when it reaches C the rays are reflected back in the direction of incidence, and $p_2 = p_1$. As A gradually moves up towards the principal focus, p_2 becomes greater than p_1, and when A coincides with the principal focus, p_2 is infinite; directly A falls within the principal focus the pencil of rays from the mirror is divergent, and p_2 falls on the other side of the mirror, the focus being a virtual one. To investigate this case observe the diagram, fig. 16, corresponding to fig. 15, except that B falls at the back of the mirror, being the intersection of the reflected ray DG produced backward until it cuts the principal axis at B.

Now the exterior angle *GDA* is bisected by *DC*, the

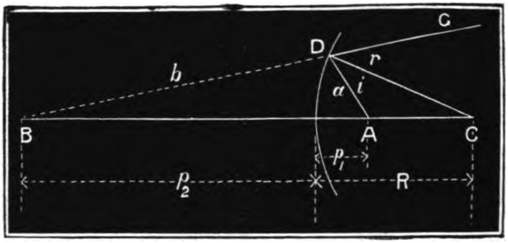


Fig. 16.

angles *i* and *r* being equal; when this is the case (see Reference Chapter A)

$$\frac{a}{b}=\frac{AC}{BC}=\frac{R-p_1}{p_2+R}$$

Assuming $a=p_1$, and $b=p_2$, we have

$$\frac{p_1}{p_2}=\frac{R-p_1}{p_2+R}, \text{ from which we have}$$

$$\frac{1}{p_1}-\frac{1}{p_2}=\frac{2}{R}=\frac{1}{f} \qquad \text{(iii.)}$$

If we make the usual convention that lines measured behind the mirrors are negative, then formula (iii.) becomes

$$\frac{1}{p_1}-\frac{1}{-p_2}=\frac{2}{R}, \text{ or } \frac{1}{p_1}+\frac{1}{p_2}=\frac{2}{R}$$

the same as formula (i.)

Now let the mirror be convex; in this case *R* falls behind the mirror, and is negative; *B* also falls behind the mirror. Hence p_2 is negative, and (see fig. 17 and Reference Chapter A)

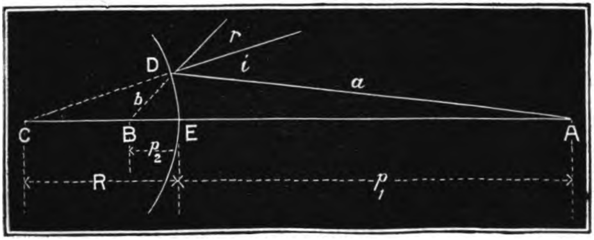


Fig. 17.

$$\frac{a}{b}=\frac{AC}{BC}=\frac{p_1+R}{R-p_2}$$

or writing p_1 for a, $-p_2$ for b, also $-R$ for R, and $-p_2$ for p_2, we have

$$\frac{p_1}{-p_2}=\frac{p_1-R}{-R+p_2},\text{ from which}$$

$$-p_1R+p_1p_2=-p_1p_2+p_2R,\text{ and}$$

$$-p_1R-p_2R=-2p_1p_2,\text{ or dividing by }-p_1R_1p_2$$

$$\frac{1}{p_2}+\frac{1}{p_1}=\frac{2}{R}$$

Hence we see that all cases of concave and convex mirrors are included in the same formula if we make the convention that lines drawn behind the mirror are negative, those in front being positive.

Abundant examples are given in *Arithmetical Physics, Part I.*, pages 20 and 68. These the student may with advantage look over before working the exercises at end of this chapter.

Relative magnitudes of object and image. Let PQ be object, and pq image, as formed by a concave mirror.

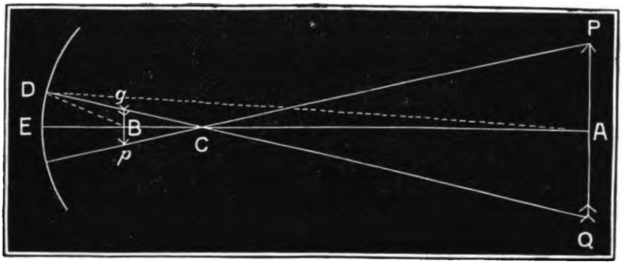


Fig. 18.

Then by similar triangles

$$\frac{PQ}{pq}=\frac{AC}{BC}=\frac{AD}{BD}$$

$$\frac{AD}{BD}=\frac{\text{Distance of object from mirror.}}{\text{Distance of image from mirror.}}$$ See page 45.

Hence

$$\frac{\text{Size of object}}{\text{Size of image}}=\frac{\text{Distance of object from mirror.}}{\text{Distance of image from mirror.}}$$

Of the exercises on curved mirrors which follow, only one, I think, requires an example.

EXAMPLE G. A point of light is placed at 24 inches from a concave mirror of 12 inches radius, and on its axis the reflected rays are received on a small convex mirror of 36 inches radius, which is placed at 6 inches from the centre of the concave mirror. To what point will the rays ultimately be reflected?

As regards the effect of the concave mirror, we find since $p_1 = 24$, and $f = 6$, that $p_2 = 8$. Hence, if the convex mirror were away, the rays would come to a focus at 8 inches from the mirror, but they are now received by the convex mirror. Hence we have to calculate the effect of this mirror upon a pencil whose point of incidence is virtual, and two inches behind the mirror; so that $p_1 = -2$, and $f = -18$ inches, from which we have, if p_2 is the distance of the reflected pencil,

$$\frac{1}{-2} + \frac{1}{p_2} = -\frac{1}{18}$$

$$\text{or } p_2 = 2\tfrac{4}{7}$$

so that the rays will come to a focus $2\frac{4}{7}$ inches from the convex mirror, that is, $3\frac{3}{7}$ inches from the centre of the concave one.

Geometrical illustration of the formula for mirrors. Rule on a drawing board two lines, *XOX'*, *YOY'* (fig. 1, plate 5), at right angles to each other. At a distance *f*, or *R*/2, from each of the lines *OX*, *OY*, mark a point *P*. Now set off on *X'OX*, to the right of *O*, a distance *OA* equal to the distance of the object from the mirror, that is, a length p_1; through *AP* draw a straight line, cutting *OY* at *B*, then the distance *OB* gives distance of image, or p_2. If the given mirror is convex instead of concave, the point *P* must be placed below *OX'*, and to the left of *OY'*, as at *P* in fig. 2 of plate 5, when the line *AP* will cut *OY'* instead of *OY*, and must be regarded as negative. Plate 5.

EXERCISE IV.

NOTE.—After the numerous examples of the application of the formula for mirrors in Part IA, the student will now only require a few exercises in the principles upon which the formula is based.

1. Rays are incident upon a mirror at the following angles, 10°, 0°, 50°. What is the angular deviation of each ray after reflection?

2. A ray reflected from a moving mirror was observed to move through 50°. Through what number of degrees had the mirror moved?

3. A ray makes an angle of incidence (i.) of 0°, (ii.) of 45° with one of two mirrors which enclose an angle of 90°. Required in each case the number of reflections in order that the reflected ray may be parallel to the incident ray.

4. Two mirrors are inclined at an angle of 20°. A ray is incident upon one of them making an angle of incidence of 70°. After two reflections, what will be the angle between the incident and reflected rays?

5. Two mirrors are inclined at an angle of 37°, and a luminous point is placed half-way between them. How many images of the point will be formed?

6. A letter *L* is placed between two mirrors inclined at an angle of 50° to each other. The upright part of the *L* is 40° from one of the mirrors, while the horizontal part extends through 3° towards the other mirror. Make a sketch of the whole series of images produced. Make also a drawing of two mirrors inclined at 72° with a letter *L* placed midway between them.

7. The mirrors of a kaleidoscope are usually of plain glass, blackened at the back. Would there be any advantage in using mirrors silvered at the back instead of blackened at the back? Give a reason for your reply.*

8. When a luminous point is at 50 cm. from a concave mirror, an image of the point is formed at 12·5 cm. in front of the mirror. What is the radius of the mirror?

9. With a concave mirror a luminous point at a distance of 100 cm. in front of the mirror gives a virtual image of the point at a distance of 200 cm. behind the mirror. Required the radius of the mirror.

10. With a convex mirror, the principal focus of which is 10 inches, a virtual image is formed at 8 inches behind the mirror. Required the distance of the object in front of the mirror.

11. An object one inch long placed 10 inches in front of a concave mirror gave a real image half an inch long. Required the principal focus of the mirror.

12. A point of light is placed at the principal focus of a concave mirror of two feet radius. The light reflected from this mirror falls upon a convex mirror of one foot radius. Where will the focus of reflected light be situate behind the convex mirror?

13. In Question 12 the focus of reflected light has been found to be 6 inches behind the convex mirror. The light from this focus now falls upon the concave mirror, and produces a second reflected focus. Where will this focus appear, assuming the mirrors to be three feet apart measured along the line joining the centres of the mirrors?

ANSWERS.

1. ± 160°; ± 180°; ± 80°. The plus sign being used if the ray strikes the mirror from above, the minus sign if it strikes the mirror from below.

2. 25°. **3.** (i.) 1; (ii.) 2.

4. −40° or +40°, according as the angle *A* is to right or left of the point where the incident ray strikes the lower mirror.

5. 10. **8.** 20 cm. **9.** 400 cm. **10.** 40 inches.

11. 3⅓ inches. **12.** 6 inches. **13.** 16·8 inches.

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* The amount of light reflected from the front of a piece of plain glass increases very rapidly as the angle of incidence approaches 90°, that is, as the pencil is oblique. With a mercurial surface the variation in the amount of reflected light varies but little as the angle of incidence increases.

LESSON V.

Refraction.

Snell's Law. The student is already familiar with Snell's law. See *Arithmetical Physics, Part I*A, page 23. This may be expressed

$$\frac{\sin i}{\sin r}=\mu, \text{ or } \sin i=\mu \sin r, \text{ or } \sin r=\frac{\sin i}{\mu} \qquad \text{I.}$$

Where i is angle of incidence of a ray, r the angle of refraction of the same ray, and μ is the index of refraction.

Thus with glass of refractive index 1·5 for the green ray, if the angle of incidence be 30° (sin 30° = 0·500), we have

$$\sin r=\frac{\sin 30^\circ}{1\cdot5}=\frac{0\cdot500}{1\cdot5}=0\cdot\dot{3}$$

and referring to a table of sines we find that the angle r is a little over 19°. It is obvious that with equation I., having given any two of the quantities, we can find the third.

Absolute index of refraction. When a ray is incident from a vacuum into a medium, the index of refraction μ expressing the relation of the sines of incidence and refraction is called the absolute index of refraction. When from a medium A to a medium B the relation of the sines is termed the index from A to B, and may be expressed as μ_{A-B}.

Interchangeableness of incident and subsequent ray. It is a universal proposition in optics founded on experiment that incident rays and the subsequent rays are always interchangeable.* Thus, if a ray of light is incident upon a medium at an angle i, and when it enters the medium it makes an angle r, then we may safely say that if it is incident *from* the medium at an angle r, it will emerge at an angle i; hence if the index of refraction from a rare to a dense medium be μ, the index of refraction from the dense to the rare medium will be $1/\mu$. For in tracing the ray back we have to find the relation $\frac{\sin r}{\sin i}$, but it is obvious since

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* Various experiments may be made to prove the universal proposition. The following, given in Young's *Natural Philosophy*, is, perhaps, the simplest:—Stand before a looking glass, close the right eye. The rays which proceed from the eyelid of the closed eye proceed to the glass, are reflected, and fall upon the open (left) eye. Place the finger on the glass so as to blot out the image of the closed eye, and now close the left eye and open the right, when it will be found that the image of the closed left eye is blotted out. This shows that whether the rays proceed from the right eye and fall on the left, or proceed from the left and fall on the right; in either case is their course the same. If the looking glass be immersed in water the same effect is observed; hence the proposition is true both for reflection and refraction.

$$\frac{\sin i}{\sin r} = \mu, \text{ that } \frac{\sin r}{\sin i} = \frac{1}{\mu}$$

Media with parallel sides. Fig. 19 represents a slab of glass with parallel sides placed in vacuo. A ray of light AB is incident on the surface at an angle i, and in the glass makes the angle of refraction r. On passing from the glass to vacuo again it is obvious, bearing in mind the universal proposition p, that since the normals at B and C are parallel, that the angle of incidence is r, and the angle of refraction must be i; that is, the rays AB and CD are parallel.

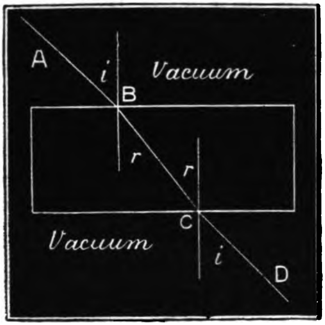

Fig. 19.

Let M, N, O, fig. 20, represent slabs of *different materials* with parallel sides. and placed parallel to each other in vacuo. Then by what precedes the rays AB and CD are parallel and it also follows that EF is parallel to CD, and that GH is again parallel to EF. But if GH is parallel to EF, and EF to CD, and CD to AB, it follows (Reference Chapter A) that GH is parallel to AB. Hence we see that with a series of slabs thus placed a ray of light incident on the first emerges at the last parallel to the incident ray.

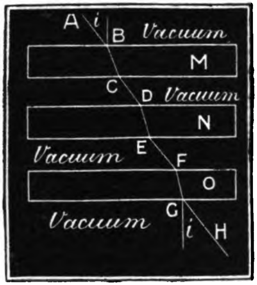

Fig. 20.

Now let the slabs move nearer and nearer together, this makes no difference as regards parallelism, and when they touch we can see no reason why any change should take place. Hence *in a series of media with parallel sides, and touching each other, a ray incident at any angle i emerges at the same angle after passing the various media in succession.**

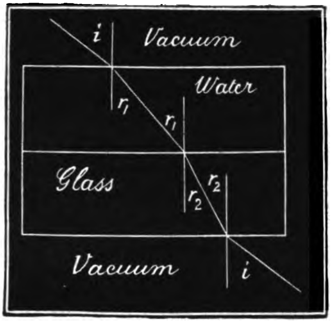

Fig. 21.

In fig 21 is represented a ray passing from vacuo through water and glass again into vacuo. From what precedes, the ray emerges parallel. Let μ_w be absolute refractive index of water, and μ_g absolute refractive index of glass, and let μ_x be index of refraction from water into glass, then

$$\frac{\sin i}{\sin r_1} = \mu_w$$

$$\frac{\sin r_1}{\sin r_2} = \mu_x$$

$$\frac{\sin r_2}{\sin i} = \frac{1}{\mu_g}$$

Multiplying the quantities on the left together, and the quantities on the right together, by which the equality is undisturbed, we have

$$1 = \mu_w \times \mu_x \times \frac{1}{\mu_g} \text{ or } \mu_x = \frac{\mu_g}{\mu_w}$$

or index from water to glass $= \dfrac{\text{absolute index of glass}}{\text{absolute index of water}}$

EXAMPLE A. The absolute index of bisulphide of carbon (CS_2) is 1·631, and of water 1·333. What is index of refraction from bisulphide of carbon to water?

Index from CS_2 to water $= \dfrac{\text{absolute index of water}}{\text{absolute index of } CS_2} = \dfrac{1{\cdot}333}{1{\cdot}631} = 0{\cdot}817$

---

* This (italicised) proposition in the books is usually made dependent for proof on experiment; but it seems to me, as above stated, to follow, from the universal proposition of page 49, combined with the assumption (axiom?), that contact will not affect the direction of the ray in the illustration given.

Water and glass have been used in the illustration of fig. 21 as being familiar cases. Of course we could not have a vacuum above the surface of water; this circumstance does not, however, interfere with the reasoning, which is perfectly general.

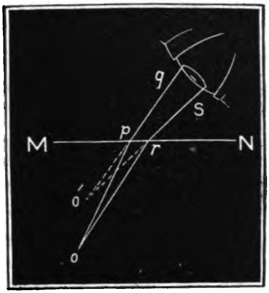

Fig. 22.

Focus of refracted rays on emergence from a plane surface. Let o be a point of light in a dense medium, say a minute portion of a small bright pebble in water, and let op, or, be a small pencil, which after refraction at pr enters the eye at qs. We have to investigate how this pencil will be affected by refraction at the surface MN so as to find the position of o', the focus of the refracted rays pq, rs, when produced backward.

Consider one ray opq, fig. 23. Draw the normal ov, and also the normal pu, and produce qp to t, as shown by the dotted line. Note that the angle vop is the same as r, the angle of refraction, and that vtp is the same as the angle of incidence i.

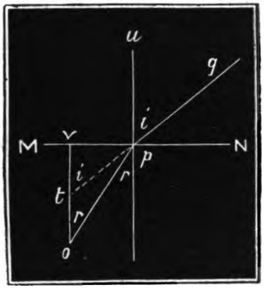

Fig. 23.

Hence

$$\sin i = \frac{vp}{pt}, \text{ and } \sin r = \frac{vp}{op}, \text{ and}$$

$$\frac{\sin i}{\sin r} = \frac{\frac{vp}{pt}}{\frac{vp}{op}} = \frac{vp}{pt} \times \frac{op}{vp} = \frac{op}{pt} = \mu$$

Hence $$op = \mu pt$$

Thus if index of refraction $\mu = 2$, we have $op = 2pt$.

To find by construction the focus of a pencil of rays proceeding from one medium to another.

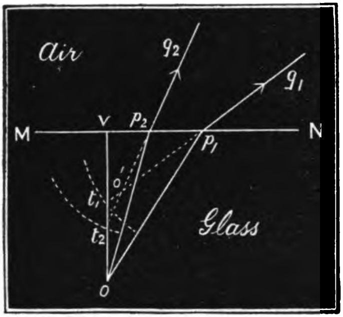

Fig. 24.

First, the pencil is from a dense to a rare medium. Let op_1, op_2, fig. 24, be the extreme rays of a pencil diverging from a dense medium into a rare one, and suppose $\mu = {}^3/_2$. From o draw the perpendicular ov. Take a length with the compasses equal to $1/\mu$ of op_1 (in this case ${}^2/_3$ of op_1), and with this radius, and p_1 as centre, strike a portion of a circle cutting the perpendicular ov in t_1, join p_1t_1 by a dotted line, and we have the course of the ray p_1q_1 in the rare medium indicated. Proceed in a similar manner with the ray op_2, making p_2t_2 two-thirds of op_2. Draw the dotted line t_2p_2, and the continuous line p_2q_2. Now to the eye receiving the refracted pencil the luminous point o will appear at o', the intersection of the two dotted lines.

Second, the pencil is from a rare to a dense medium (air to glass, for example, $\mu = {}^3/_2$).

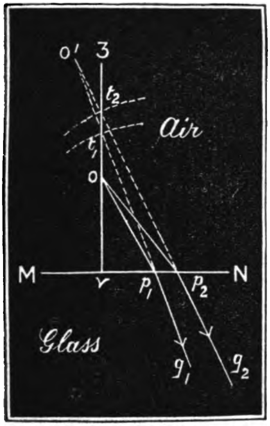

Fig. 25.

Let o be origin of pencil, and op_1, op_2, the extreme rays of the pencil. Erect the perpendicular vo, and produce it to 3. With a pair of compasses set to a radius μop_1 (in supposed case $= op_1 \times {}^3/_2$), strike a portion of circle, cutting $v3$ in t_1; draw the dotted line t_1p_1, and the continuous line p_1q_1. By similar construction find and draw the dotted line p_2t_2, and the continuous line p_2q_2; and now produce the dotted line p_1t_1, p_2t_2, to their point of intersection o', and we have the focus of the refracted pencil.

By working out a few constructions, the student will find :

A. When the eye is in the rare medium.

1. That if the eye is placed vertically above the point o, in fig. 24, the focus o' will be immediately above o, and at a distance of $ov \times 1/\mu$ from the surface.
2. That as the eye is placed either to the right or left of ov, the focus o' will move nearer to p_1p_2.

B. When the eye is in the dense medium.

1. That if the eye is placed vertically below o, that o' will be vertically above o, making $o, o' = \mu \times ov$.
2. That if the eye is placed to the right or left of ov, then o' will be farther and farther away from p_1p_2, and more and more raised from the surface.

The particular line that o' takes in fact as the eye moves in one plane to the right or left of ov, is when the eye is in a rare medium the "evolute of an ellipse," and when the eye is in a dense medium the "evolute of a hyperbola." The student will find this subject discussed mathematically in Parkinson's *Optics*, from which the following illustration (fig. 26) of the appearance of an object (an arrow) placed in a dense medium (water?) is taken.

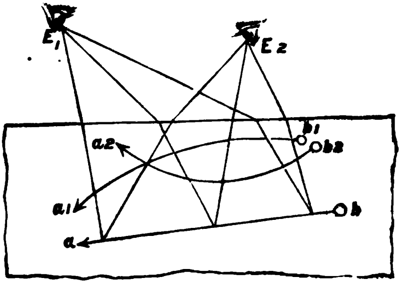

Fig. 26.

ab represents the arrow in its actual position; a_1b_1 is the appearance of the arrow to an eye placed at E_1, while a_2b_2 is the appearance of the arrow to an eye placed at E_2.

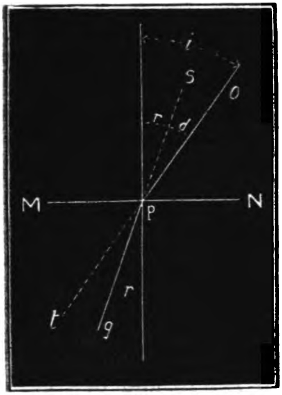

Fig. 27.

Refraction of homogeneous light through a prism of small angle. (I.) When a ray of light passes from one medium to another, the deviation is greater as the angle of incidence is greater, though not in direct proportion to it.

Thus if *op*, fig. 27, be an incident ray, and *pq* the refracted ray, the angle *ops*=*qpt* (which is the deviation)* will be greater as the angle of incidence *i* is greater. For as *op* recedes from the normal, that is, as *i* becomes greater, the angle of refraction *r*=*ops* also becomes greater; but (see foot note) the deviation is measured by the difference of angle of the incident and refracted rays, and this difference becomes greater as *i* becomes greater.

EXAMPLE B. Rays of light are incident upon glass ($\mu=1{\cdot}5$) at angles of 30°, 50°, and 80°. Required the deviation in each case.

The sine of 30° is 0·500; hence (equation I.) sin $r=0{\cdot}500\div1{\cdot}5=0{\cdot}333$, the angle corresponding to which is 19° 16′. Hence deviation of 30° is 30° − 19° 16′ = 10° 44′. Similarly the deviation for 50° will be found to be 50° − 30° 44′, or 19° 16′, and at 80° it will be 80° − 41°, or 39°. From these examples it is seen that the deviation is generally greater as the angle of incidence is greater, though not proportional to it.

When we consider the ray as passing from a dense to a rare medium, the deviation will be the difference between the directions *qps* and *tpo* (fig. 27), that is, between the angles *r* and *i* as before. Hence if a ray pass through a prism, and i_1 be angle of incidence on entrance, r_1 the angle of reflection at first face, p_2 the angle of incidence at second face, and i_2 the angle of emergence, we have the total deviation as $i_1-r_1+i_2-r_2$.

---

[*] The ray originally has the direction *opt*, but finally the direction *spq*; hence the deviation is the angle *qpt*, or *ops*, or shortly *d*. If the incident angle *i* be 20°, and the angle of refraction be 12°, then the deviation is 20° − 12°, or 8°, or generally the deviation is $i-r$.

Minimum Deviation. Let $o_1 p_1$ be a ray incident on a prism, of which *EFG* is a section at right angles to the *refracting edge* *G* of the prism. On entering the prism, the deviation by what precedes is $i_1 - r_1$, and on emergence the deviation is $i_2 - r_2$, so that for the total deviation *D* we have

$$D = i_1 - r_1 + i_2 - r_2$$
$$= (i_1 + i_2) - (r_1 + r_2)$$

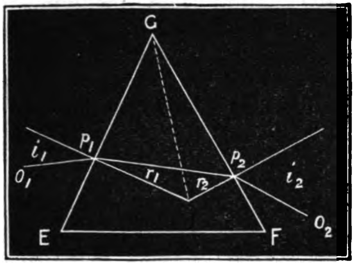

Fig. 28.

It is pretty clear that as the ray $o_1 p_1$ varies in inclination, that is, as i_1 varies, so in a general way will the deviation vary, and somewhat curiously we find that there is a *minimum deviation* when the angles of incidence and emergence are equal, that is, when $i_1 = i_2$, and of course $r_1 = r_2$.

The mode of proving mathematically that there is a minimum deviation under the above conditions is somewhat complex, but a few trials as in Problem 5, Exercise V., will soon convince the student that the statement is true. He ought also to see this point experimentally demonstrated.

EXAMPLE C. A prism has a refracting angle A of 40°, the refractive index of the material of the prism is 1·5. Find (i.) the angle of incidence that a ray must have in order to undergo minimum deviation. (ii.) Find the deviation when the angle of incidence is 40°.

From what has just been stated, and referring to fig. 29, we can see that the angle of refraction on entering the prism must be $\frac{1}{2}A$, or 20°; hence, as $\sin i = \mu \sin r$, we have

$$\sin i = 1{\cdot}5 \times \sin 20^\circ$$
$$= 1{\cdot}5 \times 0{\cdot}342 = 0{\cdot}513$$

Now referring to table of sines on cover, we find the next number lower than 0·513 is 0·500, and further that the difference between this and the next sine is 0·015, while the difference between 0·500 and 0·513 is 0·013; hence the angle corresponding to one whose sine is 0·513 is $30^\circ + (0{\cdot}013 \times 60') / 0{\cdot}015$, or 30° 52′, the required angle for minimum deviation. (ii.) We find r_1 first by the formula $\sin r_1 = \sin 40^\circ / 1{\cdot}5 = 0{\cdot}643/1{\cdot}5 = 0{\cdot}429$ nearly, from which $r_1 = 25^\circ\ 24'$.

Now as $r_1 + r_2 = A$, we have $r_2 = 40^\circ - 25^\circ\ 24' = 14^\circ\ 36'$ knowing r_2, then $\sin i_2 = 1{\cdot}5 \times \sin r_2 = 1{\cdot}5 \times 0{\cdot}252 = 0{\cdot}378$, from which we find i_2 as 22° 11′.

Now we know the value of i_1, i_2, and A, we have for the deviation $i_1 + i_2 - A$,

$$40^\circ + 22^\circ\ 11' - 40^\circ = 22^\circ\ 11'.$$

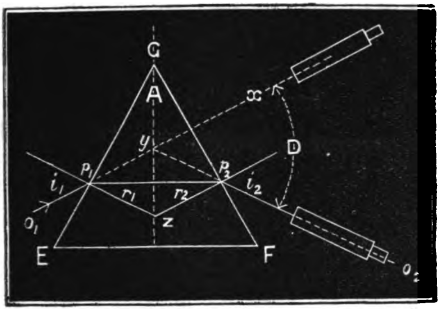

Fig. 29.

The position of minimum deviation can be found experimentally. Suppose a telescope directed to receive a small pencil of light o_1p_1x (fg. 29). Now introduce the prism *GEF* so as to deviate the light in the direction $p_1p_2o_2$. The telescope will have to be moved through the angle *D* before it will again receive the pencil of light, and on trial it will be found that by moving the prism a little either way the telescope must make a greater angle in order to keep the image on the cross wires. Now read off the angle *D*, and we have the angle of minimum deviation, and this measurement combined with a measurement of the angle *A*, which is easily made on the same instrument, enables us to find the angle of refraction of the material of the prism.

To find the index of refraction we want to know the angles i_1 and r_1.

Now (see fig. 29)

$$r_1 = \tfrac{1}{2}A \qquad *$$

$$\begin{aligned} i_1 &= yp_1z \\ &= yp_1p_2 + r_1 \\ &= \tfrac{1}{2}D + \tfrac{1}{2}A \end{aligned}$$

Hence

$$\frac{\sin i_1}{\sin r_1} = \frac{\sin \frac{1}{2}(D+A)}{\sin \frac{1}{2}A}$$

an equation from which knowing *D* and *A* the index of refraction is determined.

---

* Whatever value r_1 and r_2 may be, we have since p_1z and p_2z are perpendiculars to *GE* and *GF* respectively

$$r_1 + r_2 + Gp_1p_2 + Gp_2p_1 = 180^\circ, \text{ and}$$
$$A + Gp_1p_2 + Gp_2p_1 = 180^\circ$$

Hence $r_1 + r_2 = A$. Now if $r_1 = r_2$, as in minimum deviation,

$$r_1 = \tfrac{1}{2}A$$

† *D* is the exterior angle of the isosceles triangle p_1yp_2; hence $D = 2yp_1p_2 = 2(i_1 - r_1)$. Hence $i_1 = \frac{1}{2}D + r_1 = \frac{1}{2}D + \frac{1}{2}A$.

Condition of emergence from a prism. Suppose the refracting angle at G, fig. 29, to be so great that the ray o_1p_1 must be parallel to the side of the prism in order that it may emerge at the other side of the prism, the angles i_1 and i_2 will then each be 90°, p_1p_2 will be still parallel to the base of the prism, and r_1r_2 will each be the critical angle of the material of the prism, and together equal to A; and since there is under these circumstances minimum deviation, it follows that if the angle i is made a little less than 90° (it cannot be more), then r_2 becomes a little greater than the critical angle, and the ray is totally reflected. Hence we learn *that a ray incident upon one face of a prism can only emerge when the refracting angle of the prism is less than twice the critical angle of the material of the prism.*

EXAMPLE D. A prism has a refracting angle of 60°, and $\mu = 1{\cdot}5$ Can a ray incident at any angle on one face of the prism emerge at the other face?

The sine of the critical angle (see *Arithmetical Physics, Part IA.* page 25) is 2/3, or 0·666, from which the critical angle is found as 41° 50′ nearly, and as 60° is less than twice this value, the ray can emerge; if, however, the prism had a refracting angle of 84°, the ray would not emerge.

Law of swiftest propagation. From the principles of the wave theory of light, to be referred to hereafter (Reference Chapter C), it is shown that when light is reflected or refracted the vibration is propagated from any one point to any other in the *least possible time.* Thus, in fig. 11 the distance $AP+PR$ is the shortest distance that can be drawn from the point A to the mirror, and from thence to the point R. Again, in fig. 34 the propagation of light from A through the thickest part of the lens to the point a, and also from the point A through the thinner part of the lens at D to the same point a, is in both cases equal, and the quickest possible; the retardation of the ray through the thick part of the lens being just compensated by the longer distance the ray has to travel from A to D and D to A.

EXERCISE V.

1. Determine the relative index of refraction of each of the following bodies from the observations recorded.

Medium from which Ray is Incident.	Angle of Incidence of Sodium Light.	Medium in which Ray is Refracted.	Angle of Refraction of Sodium Light.
a. Air	50°	Water	35° 4′
b. Bisulphide of Carbon	40°	Water	51° 51′
c. Water	35°	Glass (Flint)	27° 45′
d. Air	50°	Diamond	18° 33′

2. The data of Question 1 give relative indices of refraction. Knowing that the absolute index of refraction for air is 1·00029 (sodium light), find absolute index of refraction of each of the substances named in Question 1.

3. In order to determine the index ot refraction of a glass prism, its refracting angle was measured, and found to be 48°; further, the minimum deviation was determined as 50°. From these observations find the index of refraction of the material of the prism.

4. Find the angle of incidence with which a ray must strike a prism in order to suffer minimum deviation, knowing that the angle of the prism is 30°, and that the index of refraction of the material of the prism is 2·000.

5. With the prism of Question 4 find the deviation produced (i.) when the ray enters at minimum deviation 31° 12′; (ii.) when it enters making an angle of incidence of 40°; (iii.) when it enters at 28°.

6. A ray strikes one face of a prism perpendicularly. At what angle will it emerge (i.) if the refracting angle of the prism is 30°, and the index of refraction is 1·768? (ii.) if the angle is 40°, and the index 1·333? (iii.) Will the ray emerge at the second face if the angle of the prism is 30°, and the index of refraction is 2·000?

ANSWERS.

1. (*a*) 1·330; (*b*) 0·818; (*c*) 1·213; (*d*) 2·409, all approximately.
2. Water, 1·3304; Bisulph. Carbon, 1·626; Glass, 1·614; Diamond, 2·410.
3. 1·855.
4. 31° 12′.
5. (i.) 32° 24′; (ii.) 33° 56′; (iii.) 32° 28′.
6. (i.) 62° 7′; (ii.) 59°; (iii.) the ray is just at the limit of emergence.

LESSON VI.

Refraction at a Spherical Surface. Formulæ for Lenses.

It is most convenient to consider a concave spherical surface which bounds a dense medium, a small pencil of light being incident upon such surface from a vacuum, and the axis of the pencil coinciding with the axis of the spherical surface.

Let BAB', fig. 30, be section of such a surface, the dense medium being bounded by this concave surface.

QA is a ray coinciding with the axis of the spherical surface.

Qp is any other ray proceeding from Q.

pS is the refracted ray.

pq is the refracted ray produced backward till it meets the axis.

QpO is the angle of incidence i.

qpO is equal to the angle of refraction r.

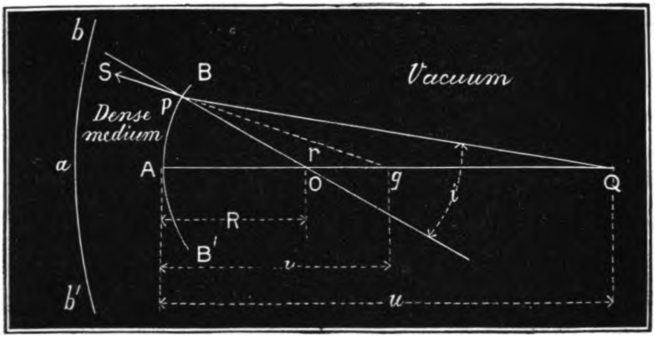

Fig. 3c.

$AQ = u$ $Aq = v$ $AO = R$, or radius of sphere.

Then $$\frac{Oq}{pq}=\frac{\sin r}{\sin pOq} \qquad \text{(i.)*}$$

$$\frac{OQ}{pQ}=\frac{\sin i}{\sin pOQ}=\frac{\sin i}{\sin pOq}=\frac{\mu \sin r}{\sin pOq} \qquad \text{(ii.)}$$

(i.) ÷ (ii.) $$\frac{Oq}{pq}\div\frac{OQ}{pQ}=\frac{\sin r}{\sin pOq}\div\frac{\mu \sin r}{\sin pOq}=\frac{1}{\mu}$$

Hence $$\mu\frac{Oq}{pq}=\frac{OQ}{pQ}$$

Now if the angle Ap is a small one, Qp may be regarded as equal to QA, that is, to u, and pq as equal to Aq, that is, to v; hence as $Oq=v-R$, and $OQ=u-R$, we have

$$\mu\frac{v-R}{v}=\frac{u-R}{u}, \text{ or } \mu\left(1-\frac{R}{v}\right)=1-\frac{R}{u}$$

Dividing both sides by R,

$$\mu\left(\frac{1}{R}-\frac{1}{v}\right)=\frac{1}{R}-\frac{1}{u}$$

that is, $$\frac{\mu}{v}-\frac{1}{u}=\frac{\mu-1}{R} \qquad (A)$$

In the above examination the light proceeds from a rare to a dense medium. If it proceed in the opposite direction, we can write $1/\mu$ for μ. (See page 52.)

If, for example, rays are proceeding from glass into vacuum, if u is distance of incident pencil, then distance w of refracted pencil will be found by substituting $1/\mu$ for μ and w for v in formula A, that is, it is

$$\frac{\frac{1}{\mu}}{w}-\frac{1}{u}=\frac{\frac{1}{\mu}-1}{S} \text{ or}$$

$$\frac{1}{\mu}\left(\frac{1}{w}-\frac{\mu}{u}\right)-\frac{1}{\mu}\left(\frac{1-\mu}{S}\right)$$

or striking out common factor $1/\mu$

$$\frac{1}{w}-\frac{\mu}{u}=\frac{1-\mu}{S}$$

S being the radius of the second surface.

---

* The sides of triangles are proportional to sines of opposite angles. (See Reference Chapter A.)

Suppose now the ray *PS*, fig. 30, to fall upon a second concave surface. *bab′* having a radius *S*, then if the lens is thin *qa* may be regarded as equal to *qA*, so that *v* remains as distance of the incident pencil, and must be substituted for *u* in last formula, so that we have for second refraction

$$\frac{1}{w}-\frac{\mu}{v}=\frac{1-\mu}{S} \qquad (B)$$

and for the first (*A*)

$$\frac{\mu}{v}-\frac{1}{u}=\frac{\mu-1}{R}$$

and adding these together to find effect of the two surfaces of the lens, we have

$$\frac{1}{w}-\frac{1}{u}=(\mu-1)\left(\frac{1}{R}-\frac{1}{S}\right) \qquad (C)$$

which is the general formula for all kinds of lenses.

In applying this formula, the student must bear in mind the way in which it has been obtained. The incident pencil *u* is always positive. and the radii *R* and *S* of the surfaces of the lens being both on the same side of the lens as *u*, are positive quantities.

With any problem the student is strongly advised to make a rough sketch, showing position of the incident pencil and the curves of the lenses; he will then easily see which lines are to be treated as positive and which negative in the formula. The following examples will, however, probably clear up any difficulty that may exist:—

Example A. Required the principal focus of a meniscus lens having radii of 8 inches and 6 inches when $\mu=1\cdot5$.

The principal focus *f* of a lens is the distance of the focus of the refracted pencil (real or virtual) from the lens when the distance of the incident pencil is infinite.

In formula *C* make *u* infinite, and we have

$\frac{1}{w}$, or what is same thing, $\frac{1}{f}=(\mu-1)\left(\frac{1}{R}-\frac{1}{S}\right)=\frac{1}{2}\left(\frac{1}{R}-\frac{1}{S}\right)$

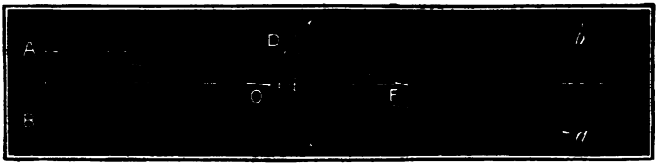

Fig. 31.

Now if a of fig. 31 is focus of incident light, then R and S, the radii of the first and second surfaces of the lens, are both positive, and

$$\frac{1}{f}=\frac{1}{2}\left(\frac{1}{8}-\frac{1}{6}\right) \text{ or } -\frac{1}{24}$$

Hence $$f=-24 \text{ inches.}$$

If, on the contrary, the convex surface is turned towards the incident pencil, then R and S are both negative; R being the radius of the first surface is now -6, while S is -8, so we have

$$\frac{1}{f}=\frac{1}{2}\left(\frac{1}{-6}-\frac{1}{-8}\right)$$

which gives $f=-24$, as before.

It will be easily seen that we may substitute $1/f$ for the expression on the right-hand of equation C. Hence

$$\frac{1}{w}-\frac{1}{u}=\frac{1}{f} \qquad (D)$$

the equation used in *Arithmetical Physics*, IA, page 24 (except that p_2 is used for w and p_1 for u), the right sign for f being determined as in the above examples.

EXAMPLE B. A double convex lens has radii of 10 and 12 cm. It is found that a point of light 15 cm. in front of the lens is refracted to a point 15 cm. on the other side of the lens. Required the index of refraction of the material of the lens.

Here $$u=15 \quad w=-15 \quad R=-10 \quad S=12$$

Hence $$\frac{1}{-15}-\frac{1}{15}=(\mu-1)\left(\frac{1}{-10}-\frac{1}{12}\right)$$

$$\mu-1=\left(-\frac{2}{15}\right)\left(-\frac{120}{22}\right)=\frac{8}{11}$$

Hence $$\mu=1+\frac{8}{11}, \text{ or } 1\cdot727$$

The focus of a series of lenses when in contact is easily found, for if u_1 is first incident focus, w_1 focus of first refracted pencil, and f_1 the principal focus of the first lens, we have

$$\frac{1}{w_1}-\frac{1}{u_1}=\frac{1}{f_1} \qquad (a)$$

Now w_1 is the focus of incident rays for the second lens, and if w_2, f_2, are lengths of refracted pencil and principal focus for the second lens, we have

$$\frac{1}{w_2}-\frac{1}{w_1}=\frac{1}{f_2} \qquad (b)$$

similarly for a third lens

$$\frac{1}{w_3}-\frac{1}{w_2}=\frac{1}{f_3} \qquad (c)$$

Now, adding *a b c* together, we have

$$\frac{1}{w_3}-\frac{1}{u_1}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}$$

so that if f is principal focus of the combination, we have

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3} \qquad (d)$$

EXAMPLE C. Two double convex lenses, one with a principal focus of 20 inches and the other with a principal focus of 10 inches, are placed in contact. What is the principal focus of the combination?

The lenses are converging. Hence f_1 and f_2 are negative.

So $$\frac{1}{f}=-\frac{1}{20}-\frac{1}{10}, \text{ or } -\frac{3}{20}, \text{ so that } f=-6{\cdot}6 \text{ inches.}$$

EXAMPLE D. A double convex lens of 20 inches principal focus is cemented to a plano-*concave* lens of 40 inches principal focus. Required the focus of the combination, and also whether the lens will be a converging or diverging one.

Here f_1 is negative, but f_2 is positive.

So $$\frac{1}{f}=-\frac{1}{20}+\frac{1}{40}=-\frac{1}{40}$$

Hence $f=-40$ inches, and the lens is a converging one as the focus is negative.

Centre of a lens. Draw o_1p_1, o_1p_3, fig. 32, *any* radii of the first surface of a lens, and respectively parallel to these radii draw o_2p_2, o_2p_4, radii of the second surface. Then as o_1p_1 is parallel to o_2p_2, the tangent at p_2 is parallel to that at p_1, and similarly the tangent at p_3 is parallel to that at p_4; and a ray of light, such as S_1p_2, which after refraction takes the course p_2p_1, will emerge parallel to its original direction, because it passes through a medium with parallel sides.

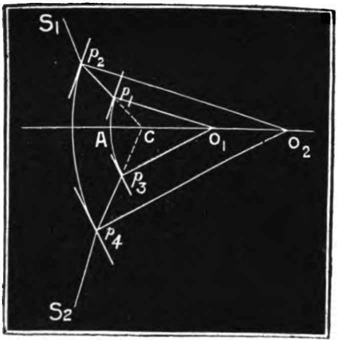

Fig. 32.

The point C, in which

the lines p_2p_1, p_4p_3, meet, may be shown to be a fixed one, and, further, it may be shown that the distance AC is equal to $rt/(s-r)$, where r and s are radii of the surface, and t is the thickness of the lens at its axis (see Parkinson, page 93; Aldis, page 82). When the lens is very thin, that is, when the surface p_2p_4 touches the surface p_1p_3 at A, all the lines p_2p_1, p_4p_3, meet on the surface of the lens; so that the centre of the lens when the lens is *very thin* is a point on the surface of the lens.

The centre of the lens has the following important optical property:—"Any ray passing through the lens in such a manner that its direction while within the lens passes through the centre will, on emerging from the lens, have a direction parallel to its direction when incident on the lens."—Aldis, page 82.

Images produced by lenses. If ABC, fig. 33, is an object in front of a lens, the ray BO coinciding with the principal axis passes through the centre of the lens, and applying the formula C, p. 64, or D, p. 65, we can find the point b to which all rays proceeding from B will be collected. The rays AO, CO, also pass through the centre, and again applying the formula, the points a and c can be found. It is obvious from the similarity of the triangles AOC, aOc, that the length AC is to the length ac as the distance OB is to the distance Ob.

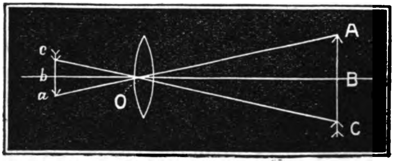

Fig. 33.

$$\frac{\text{Linear magnitude of object}}{\text{Linear magnitude of image}} = \frac{\text{Distance of object}}{\text{Distance of image}}$$

The student may with advantage show that the same is true of object and image when the image is a virtual one.

Graphic methods of illustrating conjugate foci of lenses. Fig. 34 represents a converging lens, and AB an object in front of it. We want to determine the distance *cC*, that is, the distance of the image from the centre of the lens.

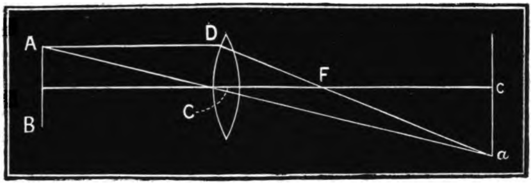

Fig. 34.

Draw *AC* through the centre of the lens, and continue the line indefinitely in the direction *AC*. Draw *AD* parallel to the axis. Set off the point *F* on the axis, making *CF* equal to *f* the principal focus. Through *DF* draw a line, and produce it until it meets the line *AC* produced in the point *a*. Draw the vertical *ac*, meeting the axis at *c*, then the distance *Cc* will be the distance of the image from the lens, if the distance *AD* is the distance of the object. Fig. 31, page 64, is a similar drawing for a meniscus lens, and fig. 35 for a diverging lens.

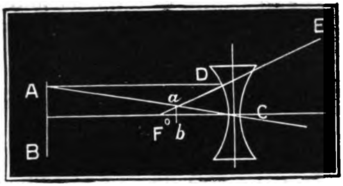

Fig. 35.

The student may reason out for himself why this construction gives position of image.

The method adopted for mirrors may also be used for lenses. Draw two lines, XX', YY' (fig. 3, plate 5), at right angles to each other, intersecting at O. From O set off a distance OA equal to the distance of the incident focus from the lens, that is, equal to u. Then if the lens is converging, set off a point P_1 at a distance f from OX, and a distance f to the right of OY^1. Through AP^1 draw a line cutting OY^1 at a_1, then the distance Oa_1 is distance of the focus of refracted light, that is, the distance w.

For a diverging lens, set off the point P_2 at a distance $-f$; f when the distance Oa_2 gives distance of focus of refracted rays, that is, gives w.

The student should trace out a few cases for himself, and see that the results agree with the formula.

EXERCISE VI.

1. Find the principal focus of a diverging lens, the radii of curvature of which are 20 and 50 cm. respectively, the refractive index of the material of the lens being 2.

2. If the above lens were immersed as an imaginary medium of refractive index 4, what would be its principal focus for parallel rays falling from the dense medium on to the lens?

3. A double convex lens, having a principal focus of 4 inches, is cemented to a diverging meniscus lens of 8 inches focal length. What is the principal focus of the combination?

4. A double convex lens of crown glass, $R'=12$ inch, $S'=12$ inch ($\mu=1{\cdot}5$), is cemented to a plano-concave lens of flint glass ($\mu=1{\cdot}6$), $R=12$ inch, $s=\infty$. What is the principal focus of the combination?

NOTE.—Determine principal focus of each, and then by method on page 66 determine focus of combination.

5. A double concave lens, made by enclosing two watch glasses (with their convex surfaces towards each other) in a metal frame, is used to see objects under water. To a person whose distance of distinct vision is 10 inches, find at what distance from the eye an object must be placed in water so as to be seen distinctly with such a lens, the radii of surfaces of the lens being equal, and each 4 inches. μ for water $=4/3$.

6. A double convex lens, whose principal focus is 5 inches, is used as a magnifier by a person whose distance of distinct vision is 10 inches. At what distance from the lens would such a person hold an object when viewing it?

7. A candle flame 4 cm. high is placed 240 cm. in front of a convex lens of 40 cm. principal focus. At what distance behind the lens is the image; and what is its height? (F 47.)

8. Taking the mean diameter of the sun as subtended at the earth as 32 minutes, what would be the diameter of his image when produced by a convex lens of 30 cm. principal focus? Tan 16 minutes $=0{\cdot}0046$.

9. An object and its image produced by a convex lens are 100 centimetres apart, and the lens is 50 cm. from the object. What is the length of the principal focus of the lens? Show by taking u more or less than 50 that $u+w$ will be more than 100.*

ANSWERS.

1. $33{\cdot}\dot{3}$. **2.** $-66{\cdot}\dot{6}$. **3.** -8 inches. **4.** -30 inches.
5. 4·4 inches. Find $f-8$ inches. Then as distance of image is to be 10 inches, make $w=10$ in the usual formula.
6. $3{\cdot}\dot{3}$ inches. **7.** 48 cm.; 0·8 cm.
8. 2·76 mm. See fig. 33, page 67. *ac* may represent image of sun. Knowing the length from *b* to centre of lens, and the angle subtended by the image, we find the length *cb* by help of a tangent table, and doubling this we have length *ac*.
9. -25 cm.

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* This method may conveniently be adopted for finding the focus of a lens. Move object and screen about until, when a distinct image is formed on the screen, the distance between object and image is the least that can be found. Measure this distance, and take one-fourth of it, which will give the principal focus of the lens.

LESSON VII.

Dispersion. The Achromatic Lens.

ABCD of fig. 36 represents a slab of a dense medium, such as glass. On the surface *AB* three rays of white light are incident, one *PQ* is perpendicular to *AB*, while P_1Q and P_2Q make different angles of incidence, the angles both of incidence and refraction being measured by the divided circle P_1P_2, of which the centre is *Q*.

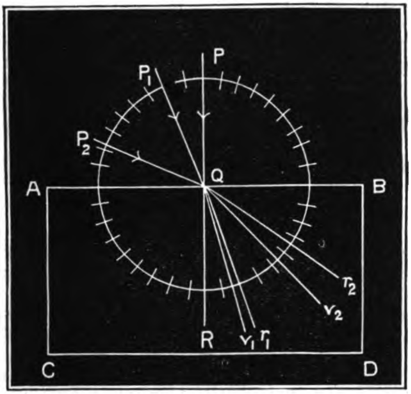

Fig. 36.

The ray *PQ* incident perpendicularly will pursue its course without material change of any kind. The ray P_1Q however is broken up into the well-known series of coloured rays, the violet ray being most and the red ray the least refracted. This separation is indicated in the figure by the letters v_1r_1 referring to the extreme rays. Similarly v_2r_2 refer to the extreme rays into which the ray P_2Q is separated. Now the angle between the extreme rays v_1r_1, or v_2r_2, is the *dispersion* of the rays by the medium *ABCD*, and it may be shown (see Lloyd, *Elements of Optics*, 1849, page 57) that the spreading out of the colours or the *dispersion* (not dispersive power) varies with the tangent of the angle of refraction. Thus in the above case the angle of incidence is 20°, and the angle of refraction (if $\mu = 1\cdot5$) is 13° nearly (page 51), and we will suppose the fan of colour to be 1°. Now to find out what will be the spread of the fan, or the dispersion at the incidence of 70°, we first find the angle of refraction, viz., 41° nearly; and taking the tangent of this 0·869, and dividing it by the tangent of 13°, or (0·231), we have 3·7 about, so the dispersion with incident angle of 70° will be 3·7 times what it was with incident angle at 20°, that is, it will be 3·7°.

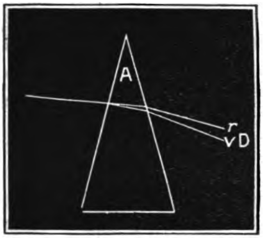

Fig. 37.

When a ray passes nearly perpendicularly through a thin prism, then dispersion takes place when the ray enters the prism, and again on leaving it (see fig. 37), and it may be shown that this dispersion, that is, the angle *D* in fig. 37, is equal to the difference of the refractive indices of the red and violet rays multiplied into the angle *A* of the prism.

EXAMPLE A. A ray of light is incident nearly perpendicularly upon a prism of flint glass having a refracting angle of 10°. The index of refraction of the violet ray is 1·671, of the red ray 1·628. Required the angular separation of the two rays as they leave the prism.

The separation $=(1{\cdot}671-1{\cdot}628)\times 10^\circ=0{\cdot}043\times 10^\circ=26'$ nearly.

Achromatic Lens. Dispersion itself, as we have seen, is simply the angle between the extreme coloured rays of the spectrum, no matter what may be the angle of incidence or other circumstances. Now this power of spreading out white light into a fan of colour, when angle of incidence is the same, varies in different substances, as is easily seen by observing the index of refraction of the violet and red ray in different media. (See table on page 75.)

EXAMPLE B. A prism of crown glass having a refracting angle of 10° has a ray of light incident nearly perpendicularly upon it. Required the angular separation of the red and violet rays, having given that the index of refraction in crown glass for the violet ray is 1·545, and for the red ray 1·524.

The separation is $(1{\cdot}545-1{\cdot}524)\times 10^\circ$, or $13'$ nearly.

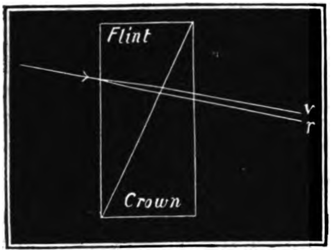

Fig. 38.

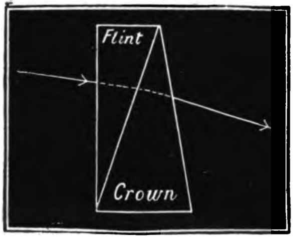

Fig. 39.

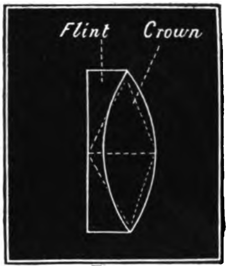

Fig. 40.

From Examples A and B we see, then, that with a prism of 10° the separation of the red and violet rays in flint glass is 26′, while in crown glass it is 13′. Let the two prisms be placed in opposite directions (fig. 38), and it will be seen that while the mean direction of the emergent ray is parallel to the incident ray, there is a slight separation of the rays, a separation actually of 26′ − 13′, or 13′.

Now let the flint prism have its refracting angle reduced to 5°, so that $0{\cdot}043 \times 5° = 13'$. (See Example A.) We shall now have the dispersion the same in both prisms, so the emergent light will be colourless, but it will no longer be parallel to the incident ray (fig. 39).

NOTE.—We have here the key to the explanation of the achromatic lens, for the achromatic lens, as in fig. 40, may be regarded as made up of four prisms—two of flint, and two of crown, the flint prisms, as indicated by dotted lines in the figure, being placed edge to edge, while the crown prisms are placed base to base.

Dispersive Power. *Achromatic Lens, continued.* The power of different substances to disperse light when the angle of a prism of a substance is small, is given by the expression

$$\omega = \frac{\mu_v - \mu_r}{\mu - 1}$$

where

ω = dispersive power of medium.

μ_v = index of refraction of violet ray.
μ_r = " " red "
μ = " " mean "

Thus the dispersive power of flint glass is found to be 0·067, and of crown glass 0·039. For with flint glass $\mu_v = 1·671$, and $\mu_r = 1·628$, while $\mu = 1·642$; and in this case $(\mu_v - \mu_r) / (\mu - 1) = 0·067$. With crown glass $\mu_v = 1·545$, $\mu_r = 1·524$, $\mu = 1·531$, and $(\mu_v - \mu_r) / (\mu - 1) = 0·039$.

Turning back to page 64, we find the general formula for a lens, where f is principal focus, μ the refractive index, and R, S, the radii of the two curves of the lens, to be

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R} - \frac{1}{S}\right)$$

Hence if f_{v_1} be principal focus of violet rays, and f_{r_1} be principal focus of red rays, while μ_{v_1} and μ_{r_1} be indices of the two rays for the substance, then

$$\frac{1}{f_{v_1}} = (\mu_{v_1} - 1)\left(\frac{1}{R} - \frac{1}{S}\right) \qquad \text{(i.)}$$

$$\frac{1}{f_{r_1}} = (\mu_{r_1} - 1)\left(\frac{1}{R} - \frac{1}{S}\right) \qquad \text{(ii.)}$$

Hence (i.) – (ii.)

$$\frac{1}{f_{v_1}} - \frac{1}{f_{r_1}} = (\mu_{v_1} - \mu_{r_1})\left(\frac{1}{R} - \frac{1}{S}\right)$$

$$= \frac{\mu_{v_1} - \mu_{r_1}}{\mu - 1} \times (\mu - 1)\left(\frac{1}{R} - \frac{1}{S}\right)$$

$$= \omega_1 \times (\mu - 1)\left(\frac{1}{R} - \frac{1}{S}\right) \qquad \text{(iii.)}$$

Since $\frac{\mu_{v_1} - \mu_{r_1}}{\mu - 1} = \omega_1$ = dispersive power.

Now, if μ = index for mean (green ?) ray, and f_1 = principal focus, for this ray we have

$$(\mu - 1)\left(\frac{1}{R} - \frac{1}{S}\right) = \frac{1}{f_1}$$

and (iii.) becomes, substituting $\frac{1}{f_1}$ for $(\mu - 1)\left(\frac{1}{R} - \frac{1}{S}\right)$

$$\frac{1}{f_{v_1}} - \frac{1}{f_{r_1}} = \omega_1 \times \frac{1}{f_1} = \frac{\omega_1}{f_1} \qquad \text{(iv.)}$$

Similarly for a second lens of dispersive power ω_2, and indices μ_{v_2}, μ_{r_2},

$$\frac{1}{f_{v_2}} - \frac{1}{f_{r_2}} = \frac{\omega_2}{f_2} \qquad \text{(v.)}$$

Now, if the second lens have the focus for violet rays the same as the focus of the first lens for red rays, and *vice versa*, that is, if $f_{v_1} = f_{r_2}$ and $f_{v_2} = f_{r_1}$, then adding (iv.) and (v.), we have

$$0 = \frac{\omega_1}{f_1} \times \frac{\omega_2}{f_2}, \text{ or } \frac{\omega_1}{f_1} = -\frac{\omega_2}{f_2} \qquad \text{(vi.)}$$

Hence if we wish to make a lens achromatic, all we have to do is to combine two lenses, positive and negative, so that if f is the focus of the mean ray we shall have (see equation *d*, page 66)

$$\frac{1}{f} = \frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} \qquad \text{(vii.)}$$

EXAMPLE C. A flint glass plano-concave lens has a principal focus for the mean (green) ray of ten inches. Required what focus for mean ray a convex crown glass lens must have for the combination to be achromatic.

From equation (vi.)

$\frac{\omega_1}{f_1} = -\frac{\omega_2}{f_2}$ and $\omega_1 = 0{\cdot}067$, $\omega_2 = 0{\cdot}039$, and $f_1 = 10$ inches; hence

$$\frac{0{\cdot}067}{10} = \frac{-0{\cdot}039}{f_2}, \text{ from which } f_2 = -5{\cdot}82 \text{ inches.}$$

EXAMPLE D. What will be principal focus for mean ray of the above combination?

By method of Example D, page 66, we have

$$\frac{1}{f} = \frac{1}{10} - \frac{1}{5{\cdot}82}, \text{ and } f = -13{\cdot}9 \text{ inches.}$$

In the above remarks reference has been made only to the extreme rays of the spectrum and the mean ray; but the colours, as given by Newton, have no definiteness, and the Fraunhofer or other fixed lines of the spectrum, with which the student is probably familiar, are invariably used in determining indices of refraction.

Newton's colours were red, orange, yellow, green, blue, indigo, and violet, the order of which may be remembered by making the initials of the colours into a barbarous word *royg-biv.** The relation of these colours to the principal

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* The usual mnemonic is the word *vibgyor*, but the inverse form of the word has the advantage that the colours are in the same order as the alphabetical series of letters indicating the Fraunhofer lines.

Fraunhofer lines of the spectrum are indicated in fig. 41, in which is represented the wide spectrum obtained from oil of cassia, and underneath the comparatively short spectrum from flint glass. These spectra are both from prisms of equal refracting angle, and illustrate well the relative dispersive powers of the two bodies. The figure is slightly modified from one given in Miller's *Chemical Physics.*

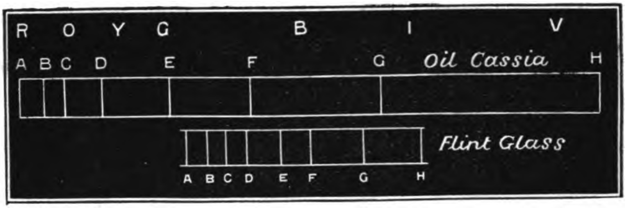

Fig. 41.

Irrationality of the Spectrum. Herewith is given a table showing

INDEX OF REFRACTION FOR DIFFERENTLY COLOURED RAYS (FRAUNHOFER'S LINES) OF THE SPECTRUM.

MEDIUM.	Red. B.	Red. C.	Yellow. D.	Green. E.	Green F.	Violet. G.	Violet. H.
Flint Glass	1·628	1·630	1·635	1·642	1·648	1·660	1·671
Crown Glass ...	1·526	1·527	1·530	1·533	1·536	1·542	1·547
Water	1·331	1·332	1·334	1·336	1·338	1·341	1·344

It will be noticed that the dispersion of the *B* and *H* lines in flint glass is 0·043 (1·671 – 1·628), while in crown glass it is 0·021 (1·547 – 1·526), or about one-half. Now with flint glass the dispersion between the *C* and *D* lines is 0·005, while in crown glass it is 0·003, that is much more than one-half. This difference in the spreading out of similar parts of the spectrum by different bodies is called ***irrationality.***

EXERCISE VII.

1. A ray of white light is incident upon a slab of glass $\mu=1{\cdot}5$ at an angle of 40°, when the angle between the extreme coloured rays is found to be 20′. If another ray fall at an angle of 75°, find, with help of table of tangents on cover, the dispersion which takes place.

2. A ray of white light is incident nearly perpendicularly upon a prism of crown glass having a refracting angle of 12°. Find the angular separation of the rays *B* and *H* on emergence. (See table, page 75.)

3. From table on page 75 find the dispersive power of flint glass and of water, taking *B* and *H* as the extreme rays, and *E* as the mean ray.

4. Find the dispersion between the rays *B* and *H* when a ray of white light falls nearly perpendicularly upon a water prism having a refracting angle of 50°.

5. A flint glass prism has an angle of 20°, and is cemented to a crown glass prism of the same angle, but with its refracting edge turned in the opposite direction. A ray falls nearly perpendicularly upon the combination. What dispersion of the rays *B* and *H* will be produced?

6. A flint glass prism and a water prism are to be combined, so that a ray of light in passing through shall not suffer dispersion, so far, at any rate, as the lines *B* and *H* are concerned. If the refracting angle of the flint glass prism is 20°, what must be the refracting angle of the water prism?

7. A flint glass concave lens of principal focus of 20 cm. is to be combined with a convex crown glass lens, so that the combination shall be achromatic so far as the rays *B* and *H* are concerned. Find the focus of the crown glass lens, and also the focus of combination.

8. A trough *ABCD*, fig. 42, has a partition *BC* diagonally across it. The under part of the partition contains bisulphide of carbon dispersion (that is, $\mu_H-\mu_B$)$=0{\cdot}084$, the upper part contains water $\mu_H-\mu_B=0{\cdot}013$. A ray of light *PQ* is incident nearly perpendicularly upon the side *AC* of the trough. Find the dispersion of the arrangement, the angle $ACB=CBD=15°$ (neglect effect of the partition).

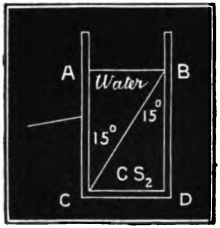

Fig. 42.

ANSWERS.

1. 35·6′. **2.** 15·12′. **3.** 0·067 and 0·039. **4.** 39′.
5. 0·44°, or 26·4′. **6.** 66° 9′. **7.** 11·6 cm.; 27·6 cm.
8. 1·065°, blue ray lowest.

LESSON VIII.

Interference. Newton's Rings.

IF two equal waves are propagated so as to interfere with each other, then if they have a difference of phase equal to half a wave length, or, indeed, any odd number of half wave lengths, they will mutually destroy each other; whereas if the difference of phase be 2, 4, 6, or any even number of half wave lengths, that is, any whole number of wave lengths, they will augment each other, and produce a wave of double amplitude. (See Reference Chapter C.)

Imagine a step-like piece of glass *B*, as in fig. 1, plate 6, placed near (commonly speaking in contact with) a piece of plane glass *A*, and let red light fall from above on to the glass. Consider what occurs on the supposition that the depths of the successive steps are ¼, ½, ¾, &c., of a wave length of red light. Plate 6.

A ray which falls upon *cd* where the surfaces are very near breaks up into a ray reflected from the surface *ab*, which will be considered later on; and another ray which passes over the minute interval at *cd* is broken up into two rays—one direct, the other reflected; the direct ray, represented by a thick line, pursues its course through the lower glass, while the other ray reflected from *cd*, and again from *ab*, also passes through the lower glass. Owing, however, to the minute interval separating the glasses, the two waves which pass through—the one represented by a thick line, and the other by a thin one—are practically in the same phase, and hence augment each other. With a ray which falls upon *ef*, separated from *ab* by a quarter wave, the case is different; for while the direct ray will have reached the position represented by the thick line, the ray reflected from *ef* is half a wave behind, and hence in a position to destroy the direct wave.* Again, the wave which falls on *gh* becomes divided at the surface *gh* into a direct wave travelling onwards, and a reflected wave which, owing to its reflection in succession by the two surfaces *gh* and *ab*, is a wave length behind the direct ray, and hence augments it. Suppose a beam of red light to fall from above, then to the eye placed underneath the step-like glass the step *cd* will have an excess of red, *ef* will have deficiency of red, while *gh* will have an excess of red. If the medium were such that the light incident on the successive steps could be broken up into a transmitted beam, and a twice reflected beam, in such a manner that the intensity of the twice reflected beam was equal to that of the directly transmitted beam, then we should have, to the eye placed below the glass, complete darkness at *ef*, and intense redness at *cd* and *gh*.

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* To convey the idea that the direct wave is in advance of the reflected wave, the thick line representing the direct wave is drawn half a wave forward.

So far we have only traced the rays which pass *through* both glasses. Consider now the light reflected upward from the surface *ab*, and the other surfaces *cd*, *ef*, &c. (See fig. 2, plate 6.)

The wave 1 on reaching the surface *ab* is reflected positively (see Reference Chapter B), that is, the reflected wave starts back without loss, as shown by the thin continuous line near. That part of the wave 1 which falls upon the surface *cd* is reflected negatively, as shown by the dotted line, losing half an undulation. Now it will be noticed that as the interval between *ab* and *cd* is minute. the ray reflected from the surface *ab*, and that reflected from *cd* are in opposite phases, and start practically together, hence destroy each other. The ray 2, similarly, is reflected positively at the surface *ab*, as shown by the continuous thin line, and negatively at *ef*, as shown by the dotted line. Now the wave reflected at *ab* will have advanced half a wave before the reflection from *ef* reaches the surface *ab*; hence the two reflected waves represented by the thin continuous line and the dotted line are, since the interval between *ab* and *ef* is quarter of a wave, just in coincidence as regards phase, and, consequently, augment each other. It may easily be seen without further description that the wave 3 produces two upwardly reflected rays which, from the retardation of a wave length owing to the half wave space between *ab* and *gh*, are thrown in opposite phases, and destroy each other.

If the red light is used and the eye is placed above the glass *A*, then the step *cd* will appear dark, the step *ef* bright, *gh* will be dark, and so on.

In the explanation given above, the light is supposed to fall perpendicularly upon the air space, separating the glasses. If the light fall obliquely, a little consideration will show that the light reflected from the two surfaces bounding the air space has a longer journey to make before emergence; hence the difference of phase of the interfering rays will be altered. Thus, when the glasses are actually a quarter wave apart, and a dark interference is produced on emergence at the step *ef* by a perpendicular ray, the case will be altered if the incident light is oblique enough to make the path of the ray reflected from *ef* twice as long, for then the difference of phase will be a wave length instead of a half wave length, and the interference will produce darkness instead of brightness.

If water instead of air is between the plates, the speed of the reflected ray is different, and a different interference results.

The apparatus called Newton's Rings consists of a plano-convex lens of long radius, placed on a piece of flat glass, as represented in section at fig. 1, plate 7. When red light falls from above upon the arrangement, there is seen by reflected light a dark spot in the centre, succeeded by a ring of red light; then follow alternate rings dark and light, gradually getting nearer and nearer to each other, and at the same time narrower and narrower. If violet light is used, then the dark spot is succeeded by alternate bright and dark bands of violet light, but the dark spot is smaller, and the rings generally are narrower. As may easily be imagined, if both red and violet light fall upon the apparatus, there will be seen by reflected light a dark spot succeeded by alternate dark and bright rings, the bright rings consisting of an inner violet and an outer red ring. Further, if white light fall upon the apparatus a succession of coloured rings are produced, having generally a violet or blue cast in the narrower portion. Plate 7.

"The colours, whatever glasses be used, provided the incident light be white, always succeed each other in the very same order, that is, beginning with the central black spot, as follows:—

"**First ring**, or first order of colours, **black, very faint blue, brilliant white, yellow, orange, red.**

"**Second ring**, or second order, **dark purple, or rather violet, blue, green** (very imperfect, a yellow green), **vivid yellow, crimson red.**

"**Third ring**, or third order, **purple, blue, rich grass green, fine yellow, pink, crimson.**

"**Fourth ring**, or fourth order, **green** (dull and bluish), **pale yellowish pink, red.**

"**Fifth ring**, or fifth order, **pale bluish green, white, pink.**

"**Sixth ring**, or sixth order, **pale blue green, pale pink.**

"**Seventh ring**, or seventh order, **very pale bluish green, very pale pink.** After then the colours become so faint that they can scarcely be distinguished from white."—*Encyclopædia Metropolitana.* Article "Light," by J. F. W. Herschel, page 463.

By transmitted light a complementary series of rings is seen, but so much white light passes through that they are feeble as compared with the reflected rings. The appearance of the rings, so far as three of them are concerned, is represented in figs. 2 and 3 of plate 7.

By Newton's Rings apparatus the length of the waves of light were first determined. The calculation is simple. Measuring the diameter of one of the rings, we obtain the radius of the ring, squaring this radius and dividing by the diameter of the sphere of which the lens is a part, we have the interval between the glasses where the ring occurs. If we are measuring the first bright ring, as seen by reflected light, this interval corresponds to a quarter of the wave. If we are measuring the second bright ring, the interval between the glasses will be three-quarters of a wave, and so on. For principle upon which the calculation depends, see Reference Chapter A.

EXAMPLE A. With a Newton's Rings apparatus the diameter of the second bright ring produced by reflected light from a sodium flame was 0·230 cm. The diameter of the sphere of which the lens formed part was 300 cm. Required the wave length of sodium light from these observations.

If t = interval between the glasses where the ring occurs,
r = radius of ring,
d = diameter of sphere,

$$\text{then } t=\frac{r^2}{d}=\frac{(0{\cdot}115)^2}{300}=0{\cdot}00004408 \text{ cm.}$$

Now the thickness between the glasses at the second ring is ¾ λ where λ is length of wave.

Hence λ = 0·00004408 × 4/3 = 0·00005876 cm., or 587·6 millionths of a millimetre. Or since a millionth of a millimetre is called a *micro-millimetre*, and represented by the symbol $\mu\mu$, we may express the answer as 587·6 $\mu\mu$.

The student should notice that the interval between successive rings (bright and dark) increases a quarter wave with each ring, but that between successive bright or successive dark rings the interval is half a wave. Thus, with the first bright ring, as seen by reflection, the interval is $\frac{1}{4}\lambda$, but the interval between the glasses at the fifth bright ring is $(4 \times \frac{1}{2} + \frac{1}{4})\lambda$, that is, $(9/4)\lambda$, and generally the interval at the *n*th bright ring is $\frac{2n-1}{4}\lambda$.

"The principal laws of these phenomena (Newton's Rings) are included in the following propositions:—

(i.) "In homogeneous light the rings are alternately *bright* and *black;* the thicknesses corresponding to the bright rings of succeeding orders being as the *odd* numbers of the natural series, and those corresponding to the black rings as the intermediate *even* numbers.

(ii.) "The thickness corresponding to the ring of any given order varies with the colour of the light, being greatest in red light, least in violet, and of intermediate magnitude in light of intermediate refrangibility.

(iii.) "The thickness corresponding to any given ring varies with the *obliquity* of the incident light, being very nearly proportional to the secant* of the angle of incidence.

(iv.) "The thickness varies with the *substance* of the reflecting plate and in the inverse order of its refractive index."—Lloyd, *Elementary Treatise on the Wave Theory of Light*, 1857, page 102.

The fourth proposition may be illustrated by introducing water, oil, or other liquid, between the two glasses. The rings contract more and more, as the liquid has a higher refractive index.

Newton measured the rings by compasses. The more accurate method, as described in Jamin's *Cours de Physique*, III, 1887, page 587, is to place the Newton's Rings apparatus on a table which can be moved horizontally by a micrometer screw. The cross wires of a telescope are focussed on the edge of one of the rings, and the table moved until the other edge cuts the wires. The movement of the table (supposing the centre of the ring has passed the cross wires) gives the diameter of the ring.

The wave lengths of light are, however, more generally measured with the diffraction grating and spectroscope. (See Glazebrook and Shaw, *Practical Physics*, 1885, page 315, and Glazebrook, *Physical Optics*, 1883, page 148.)

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* The secant of an angle is the reciprocal of the cosine, or 1/cos.

NOTE.—MEASUREMENT OF WAVE LENGTH.—" Ångström's determinations were made with the diffraction grating. Rowland's last are also by the grating. The advantage of the grating is that a perfectly definite *image* of the slit is obtained, while with the bi-prism and with Newton's Rings it is a case of maximum and minimum brightness, no definite image being obtained."

"For measurement of Newton's Rings the plan adopted at the Normal School of Science, South Kensington, is to incline a strip of plate glass at an angle of 45° to the axis of a reading microscope. The lens is then secured in position by three dabs of wax. The microscope moves over a brass slide graduated into millimetres, and provided with a vernier reading to 0·1 mm., and the breadth of the rings is read, thus avoiding foreshortening due to the tilting of the glass and lens. The method is generally used as a comparative one, sodium light being taken as one of known wave length."—A. E. Briscoe.

EXERCISE VIII.

1. On a globe of three feet diameter two concentric small circles of half an inch and one inch radius are described. The globe is then put on a flat surface, so that the small circles are at every point at an equal distance from the flat surface. Find what this distance is in the case of each ring.

2. The top of an object fifty feet high can just be seen above the surface of the still water of a lake when the eye is level with the surface of the water. What is the distance of the object from the eye? Assume the diameter of the earth as 8000 miles.

3. In one of Newton's experiments he found the radius of the fifth ring to be 8/79ths of an inch, while the diameter of the sphere of which the lens formed part was 182 inches. From these data find thickness of air film at the fifth ring, and also thickness of air film at the first ring.

4. The second bright ring with Newton's apparatus and reflected light was found to have a diameter of three mm., and the diameter of the sphere of which the lens formed part was five metres. Find λ.

ANSWERS.

1. 1/144th inch; 1/36th inch.
2. 45,960 feet.
3. 16/268363rd inch; one-fifth of this.
4. 600 millionths of a millimetre, or 600 $\mu\mu$.

HEAT.

LESSON IX.

Expansion. Correction of Thermometer and Barometer Reading. The Gridiron Pendulum.

THE mode of using coefficients of expansion in simple cases has already been treated (*Arithmetical Physics, Part IA*, page 29). The following remarks and exercises must be regarded as a continuation of the subject:—

The linear expansion of solids, as will be seen by reference to text books, is determined by direct measurement of the increase in length of a bar of the solid.

Cubical expansion may be determined from the linear by methods already explained; it may, however, be obtained directly by weighing the solid at two different temperatures in a liquid, the absolute expansion of which is known.

Suppose, for sake of illustration, that a solid when weighed in water at 4°C. loses 10,000 grammes. We know from this that the bulk of the solid at 4°C. is 10,000 cubic centimetres. Let the solid be now weighed in water at 40°C., at which temperature we know, from researches to be referred to presently, that each cubic centimetre has a mass of 0·9923 gm., and suppose in this warm water the solid which now has a temperature of 40° loses 9940 gm. Since each cubic centimetre of water displaced at the higher temperature = 0·9923 gm., the bulk of the solid in cubic centimetres will be 9940/0·9923 = 10,017·13 c.cm., that is, 10,000 c.cm. of the solid at 4° have become 10,017·13 c.cm. at 40°, or there has been an increase of 17·13/10000ths in the bulk of the solid when raised through (40 – 4), or 36°C., or an increase of 17·13/(10,000 × 36), or 0·000048 for 1°, that is, the cubic coefficient of expansion of the solid is 0 000048.

Of course any method by which the specific gravity of a body can be determined at two different temperatures serves to determine its expansion, for the bulk of a body is known when its specific gravity is known.

EXAMPLE A. A solid which has a specific gravity of 5 at 0°C. has a specific gravity 4·9 at 100°C. Required its cubical expansion.

Bulk of 5 grammes of the solid at 0°C. is one cubic centimetre (*Arithmetical Chemistry*, *Part I.*, Lesson II.), while bulk of 4·9 grammes at 100° is also 1 c.cm. Now to find the bulk of 5 grammes of the solid at 100°C., we have

$$\begin{aligned} 4{\cdot}9 \text{ gm. at } 100^\circ &= 1 \text{ c.cm.} \\ 5{\cdot}0 \;\;,, \quad\;\; ,, \;\; &= x \end{aligned}$$

and $x = 50/49$, or 1·0204 c.cm. Hence 5 grammes or 1 c cm. of the solid at 0° becomes 1·0204 c.cm at 100°, from which we find expansion for 1°C. equals 0·000204.

Absolute dilatation of liquids. If we have one cubic centimetre of a liquid at t_1°, and it becomes $(1+a)$ cubic centimetre at t_2°, then a is the *absolute dilatation* of the liquid for the range $t_2 - t_1$ degrees. The *mean absolute dilatation* for each degree of the range is, of course, $a/(t_2 - t_1)$.

The *coefficient of absolute dilatation* refers to cases in which the difference between t_2 and t_1 is one degree.

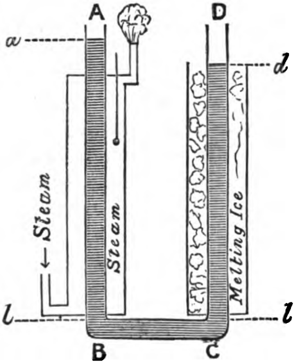

Fig. 43.

The U-shaped glass tube $ABCD$ (fig. 43) contains mercury. AB is surrounded by water at various temperatures or by steam, while CD is surrounded by ice-cold water, the horizontal portion of the tube is neglected, as a variation of temperature along this portion of the tube will not affect the result. Since the two tubes containing mercury at different temperatures are connected by the tube BC, their pressures must be equal at ll; hence the mass of mercury in AB above the level ll will, if we suppose the tubes of uniform bore,* equal that in CD above the same level. Suppose each tube to contain 300 grammes of mercury, then the bulk of 300 grammes at 100°C. is as much greater than 300 grammes at 0° as the length al is greater than the length dl, and the ratio holds with any mass, so that if we have a mass of mercury equal to 1 cubic centimetre at 0°C., this will at 100°C. become

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* Of course the tubes need not be of uniform bore for the experiment, but it is simpler to consider them so.

$$1 \times \frac{al}{dl}$$

EXAMPLE B. With the U-shaped arrangement just described, the column of mercury dl kept at 0°C. was just 100 centimetres long, while the column in al kept at 100°C. was 101·82 centimetres long. From these data determine the mean absolute coefficient of expansion of mercury for the range 0° to 100°.

Suppose each tube to be 1 square centimetre in area, then from the question we know that 100 c.cm. of mercury at 0° will become 101·82 c.cm. at 100°, that is, 100 c.cm of mercury increase 1·82 c.cm. for 100°, or 0·0182 c.cm. for 1°, and 1 cubic centimetre will increase one-hundredth of this, or 0·000182 c.cm., and this is the mean coefficient asked for.

NOTE.—Mercury does not expand equally between 0° and 100°, the expansion increasing for each degree as the temperature rises. Thus, the mean coefficient between 0° and 10° is 0·00017925, while between 0° and 100° it is 0·00018153, and between 0° and 300° it is 0·00018658; hence with liquids, in order to attain precision, we must have as many coefficients as there are degrees in the scale. Mercury varies less than any liquid; hence one reason of its applicability to thermometers.

The absolute expansion of mercury being known, we may by its help find the capacities of vessels at different temperatures. Knowing the capacities of vessels at different temperatures, we can find the mass of known volumes of liquids at different temperatures, and this is all that is wanted to determine absolute expansion.

EXAMPLE C. A glass flask at 0°C. holds 1000 grammes of mercury at 0°C., but the same flask at 100°C. holds 999 grammes of mercury at 100°C. Required the capacity of the flask at 0°, and also at 100°C., knowing (i.) that the sp. gr. of mercury at 0° is 13·596, and (ii.) that the absolute dilatation of mercury for the range 0° to 100° is 0·0181 per cubic centimetre.

Mass of 1 c.cm. of mercury at 0° .... $= 13{\cdot}596$ grammes.

,, ,, ,, $100° = 13{\cdot}596 \times \frac{1}{1{\cdot}0181} = 13{\cdot}354$,,

The capacity of the flask at 0°C. will be 1000/13·596, or 73·551 c.cm., and at 100° it will be 999/13·354, or 74·809 c.cm.

EXAMPLE D. From data of last example determine coefficient of expansion of the glass of which the flask is made.

If the interior of the flask were filled with a solid of the same material as the flask itself, then we see from Example C that 73·551 c.cm. of the material at 0°C. would become 74·809 c.cm. at 100°C. Hence expansion of 73·551 c.cm. at 0° when heated to 100°C. = 74·809 − 73·551, or 0·258 c.cm.; and of 1 c.cm. for 100° = 0·258/73·551, or 0·0035; and of 1 c.cm. for 1°C. = 0·0035/100, or 0·000035, that is, cubic coefficient of expansion of the material is 0·000035.

In connection with this example, it may be noted that the cubical expansion of the glass is the difference between the absolute and apparent expansion of the mercury. For it is evident that if the glass expand at just the rate of the mercury, a bottle once filled with mercury will remain exactly filled at any temperature, because the reservoir expands or contracts at the same rate that the contents expand or contract. If, however, the glass on being heated to 100° expand 1 c.cm. while the mercury expands 3 c.cm., then 2 c.cm. of mercury will flow out, and 3 − 2, or 1 c.cm., expresses the difference between the two expansions or the expansion of the glass.

In Example D, the two temperatures being known, and the apparent expansion of the mercury, the expansion of the glass is determined. If we know the expansion of the glass and the absolute expansion of mercury, then the range of temperature can be determined by filling the vessel at one temperature with mercury, heating it to another unknown temperature, and weighing the mercury which remains in the vessel. This is the *weight thermometer.*

EXAMPLE E. The apparent expansion of mercury in a certain glass vessel used as a weight thermometer was found to be 0·00015 per degree. To determine the temperature t of an oven the instrument was weighed full of mercury at 0°C., when the weight was 1040 gm. It was now placed in the oven to acquire the temperature t, during which 30 gm. of mercury were expelled. The weight of the glass vessel was 40 gm. Required the temperature t of the oven.

The mass of mercury in the vessel at 0° = 1040 − 40, or 1000 grammes, and mass of mercury in the vessel at t° = 1000 − 30, or 970 grammes.

Hence we see that 970 grammes of mercury at t° occupy the volume of 1000 grammes at 0°, and assuming that weights of mercury at different temperatures are proportional to volumes that they occupy, we learn that 970 volumes of mercury at 0° expand to 1000 volumes at t°, or that the mercury increases 30/1000ths of its volume between 0° and t°, and as it expands 0·00015 for each degree, the number of degrees will be (30/1000) ÷ 0·00015, or 200; hence $t = 200$°C.

Absolute expansion of water. To determine the absolute expansion of water, the water is weighed at different temperatures in a vessel, the capacity of which is known at the different temperatures, or still more simply by constructing a large thermometer, and determining by the principles already cited the capacity of the bulb and of the stem through a range of temperature. The stem is now marked so as to indicate capacities at different temperatures, and a liquid being placed in the thermometer and warmed, we can tell directly how its volume increases with change of temperature. (Despretz's method.)

Another method is to weigh a rod of glass in water at different temperatures. The expansion of the glass is determined from its linear expansion. (Matthiessen's method.)

Expansion of gases. This has been treated at length in *Arithmetical Physics*, *Part I*A, page 32. One point not referred to may be illustrated by an example.

EXAMPLE F. A gas expands 1/373 of its volume when heated to 100°C. to 101°C. Required the increase in bulk (that is, the coefficient) if heated from 0° to 1°, knowing that the gas expands (and contracts) uniformly with equal increments of temperature.

From question the gas will *diminish* 1/373 for each degree. Hence

$$1 \text{ vol. at } 100^\circ \text{ will become } 1 - \frac{100}{373}, \text{ or } \frac{273}{373} \text{ at } 0^\circ,$$

$$\text{or} \quad \frac{273}{373} \text{ volumes at } 0^\circ \text{ will become } 1 \text{ at } 100^\circ$$

$$1 \quad ,, \quad ,, \quad ,, \quad \frac{373}{273}, \text{ or } 1 + \frac{100}{273} \text{ at } 100^\circ,$$

or $\frac{1}{273}$ for each degree. Hence its coefficient at 0° is 1/273 for 1°.

Reduction of Barometer column. A good barometer (with the construction of which, no doubt, the student is already familiar) is provided with a brass scale, which has been accurately divided into inches or millimetres at some known temperature—in England at 62° F., in France at 0°C. This scale can be moved up and down so as to bring the ivory point just in contact with the mercury in the cistern. The vernier gives the actual length on the scale of the mercury column. Suppose we wish our readings to be reduced to 0°C., and that we have a barometer with a brass scale which has been accurately divided into millimetres at 0°C. It is evident that if we make an observation with such a barometer when the temperature of the instrument is 0°C., the height, as indicated by the vernier, is the absolute length of the mercury column at 0°C. Let the instrument be now at a high temperature, to take an extreme case, say 100°; we want to find what would be the absolute height of the mercury column at 0°C. Suppose the height of the column, as shown by the vernier, is 650 mm. The mercury column is not actually 650 mm. long, but is somewhat more, owing to the fact that each millimetre on the brass scale is a little longer than before.

The linear coefficient of expansion of brass is 0·000019. Hence the 650 mm. apparent length of brass at 100° will be $650 \times (1 + 0{\cdot}000019 \times 100)$, or 651·23 real millimetres.

Having now found that the real height of the mercury column at 100°C. is 651·23 mm., we find what it is at 0°C., viz., $\frac{651{\cdot}23}{1{\cdot}018153}$, or 639·62 mm.

NOTE.—In practice the method is simplified, for it will be noticed that if a is the coefficient of expansion of the brass, and b the coefficient of expansion of the mercury, t the temperature, H the observed reading, and H^1 the corrected reading, we have first found $H(1+at)$, and then divided this result by $1+bt$, that is, we have found $H^1 = H\frac{1+at}{1+bt}$

But multiply the numerator by $1-bt$, we have $1+at-bt-abt^2$, and as the quantity abt^2 is small, it may be neglected. Again, multiply the denominator by $1-bt$, we have $1-b^2t^2$, which may be taken as equal to 1, since b^2t^2 is very small, so that

$$H^1 = H\frac{1+at}{1+bt} = H\{1+(a-b)\,t\}$$

or the correction is the observed reading less the difference of the coefficients multiplied into the observed temperature.

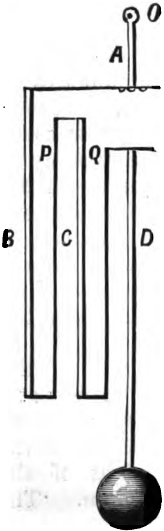

Fig. 44.

The Gridiron Pendulum. A diagrammatic representation of one-half of this pendulum is given in fig. 44. The rods A, B, C, D, are of one metal—steel, for example, while the rods P and Q are of another metal—brass, for example.

Suppose the arrangement warmed through $t°$ all the rods lengthen, but as the point O is fixed, the rods A, B, C, D, expand downwards, while the rods P and Q expand upwards.

If a_1 is the coefficient of expansion of the material of rods A, B, C, D, and a_2 is the coefficient for the rods P and Q, then

$$\begin{aligned}\text{Expansion downwards} &= a_1At + a_1Bt + a_1Ct + a_1Dt \\ &= L_1a_1t \text{ if } L_1 = A+B+C+D \\ \text{Expansion upwards} &= a_2Pt + a_2Qt \\ &= L_2a_2t \text{ if } L_2 = P+Q\end{aligned}$$

But if the pendulum is properly compensated, the expansion upwards is equal to the expansion downwards, or $L_1a_1t = L_2a_2t$, from which

$$\frac{L_1}{L_2} = \frac{a_2}{a_1} \qquad a$$

that is, the united lengths of the two sets of rods must be inversely as their coefficients of expansion. The equation *a* enables us to find any one of the quantities when the other three are given. If, however, the length L of the pendulum is given, and the lengths of the two sets of bars have to be found, we must note that $L = A + B - P + C - Q + D$, that is, $A + B + C + D - (P + Q)$, or $L_1 - L_2$.

EXAMPLE G. A gridiron pendulum is made of iron rods $a_1 = 0{\cdot}00001$, and brass rods $a_2 = 0{\cdot}00002$. If the iron rods together $= 80$ inches, what must be the united lengths of the brass rods?

From equation *a*

$$L_2 = \frac{L_1a_1}{a_2} = \frac{80 \times 0{\cdot}00001}{0{\cdot}00002} = 40 \text{ inches.}$$

EXAMPLE H. A gridiron pendulum made of iron and brass is to be 39 inches long. What must be the united lengths of the iron, and what the united lengths of the brass rods?

$$L_1 - L_2 = 39$$

But from *a* $\qquad L_2 = L_1a_1/a_2 = \frac{1}{2} L_1$

Hence $L_1 - \frac{1}{2} L_1 = 39$, and $L_1 = 78$, and $L_2 = 78 - 39$, or 39.

EXERCISE IX.

1. The specific gravity of a liquid at 0° is 2·000, at 10° it is 1·950, and at 100° it is 1·300. Find the volume of 100 grammes of the liquid at each of these temperatures.

2. Find the mean coefficient of dilatation of the above liquid between 0° and 10°, and between 0° and 100°.

3. The mean coefficient of dilatation of mercury between 0° and 350° is 0·00018784. What is the mass of 1 cubic decimetre of mercury at 350°, it being given that the mass of 1 cubic decimetre at 0° is 13,596 grammes?

4. A flask when empty weighed 40 gm.; when filled with mercury at a temperature of 10° it weighed 1055 grammes. It was now put into boiling water of temperature 100°, when mercury flowed out, and when no more mercury flowed out the flask and its contents were cooled, and the weight found to be 1040 grammes. Find coefficient of expansion of the glass.

5. A weight thermometer full of mercury at 0°C. weighed 250 grammes; empty it weighed 50 grammes. On heating it to a temperature t mercury was expelled, and on cooling it weighed 245 grammes. Find t. Take apparent expansion of mercury as 0·00015.

6. To determine the absolute expansion of a liquid, some of it was placed in a U-shaped tube, and it was found that when the temperature of the liquid in one limb was 0°, and its length 50 centimetres, the length in the other limb was 55 centimetres, and the temperature 100°. From these observations find the absolute expansion of the liquid.

7. Air has a coefficient of expansion of 1/323 at 50°C. Assuming, as is actually the case, that the expansion is uniform, required its coefficient at 0°C.

8. The linear coefficient of expansion of a metal is 0·001 at 100°. Assuming coefficient uniform, what will be the coefficient at 0°?

9. A barometer with brass scale showed an apparent height of the mercury column equal to 720 mm. when the temperature was 20°C. Find the height corrected to 0°C.

10. A gridiron pendulum is to be made of zinc and iron rods. (*a*) What must be the relative lengths of these rods? (*b*) If the length of the iron rod is 13 feet, what ought to be the length of the zinc rod? (*c*) If the pendulum itself is to be 13 feet long, what must be the lengths of the zinc, and what that of the iron rods? For iron $k = 0·001167$, for zinc $k = 0·003331$.

ANSWERS.

1. 51·28 c.cm.; 76·92 c.cm.
2. 0·00256; 0·005384.
3. 12,759·8 gm.
4. 0·00017 nearly.
5. 171° nearly.
6. 0·001.
7. 1/273.
8. 0·001, or 1/900.
9. 720·13.
10. (*a*) 1167/3331; (*b*) 4·554 feet; (*c*) 20·01 feet, 7·01 feet.

LESSON X.

Latent Heat. Dew-Point. Hygrometry.

THERE is little to add here on the subject of latent heat as far as liquids are concerned. In the examples given in *Arithmetical Physics, Part IA*, of mixtures of ice and water, the ice has always been supposed to have a temperature of 0°. Important experiments were made by M. Person (*Ann. Chim. Phys.*, [3] xxx., page 73) with ice at temperatures below 0°, and he came to the conclusion that the latent heat of water was 80 instead of 79·25, as found by Regnault.

Total heat of steam. To determine the latent heat of steam Regnault condensed known weights of steam at known temperatures and pressures in known weights of water. Many precautions were taken in order to eliminate error, and the results obtained are generally regarded as accurate. A translation of Regnault's memoir will be found at page 239 of *Chemical Reports and Memoirs*, Cavendish Society, 1848.

As a result of these experiments, Regnault obtained the total heat of steam at various temperatures and pressures. He found, for example, that if the atmospheric pressure were so diminished that water would boil at 0°,* then 1 gramme of this vapour at 0° if condensed to 1 gramme of water at 0° gave out 606·5 calories, that 1 gramme of vapour at 100°C. in condensing to 1 gramme of water at 0° gave out 637·0 calories, and that 1 gramme of vapour at 200° in condensing to 1 gramme of water at 0° gave out 667·5 calories. Hence 606·5 is the total heat of steam at 0°, 637·0 the total heat of steam at 100°, and 667·5 the total heat at 200°. The result of Regnault's research is given in the following formula:

$$Q = 606{\cdot}5 + 0{\cdot}305t$$

where Q = total heat of steam, and t = temperature of the steam.

---

* With Carré's freezing apparatus it is extremely interesting to see water boiling at the same time that spicules of ice are shooting out from the sides of the decanter.

EXAMPLE A. Required, according to Regnault's formula, (*a*) the total heat required to raise 10 grammes of water from 0° to steam at 50°; (*b*) the latent heat of steam at 50°.

(*a*) According to the formula the quantity Q for 1 gramme will be $606 \cdot 5 + 0 \cdot 305 \times 50$, or 621·75 calories, and for 10 grammes 6217·5 calories.

(*b*) Since calories required to raise 1 gramme of water from 0° to steam at 50° is by (*a*) 621·75, and to raise 1 gramme of water from 0° to water at 50° is 50, the latent heat of steam at 50° is $621 \cdot 75 - 50$, or 571·75 calories.

EXAMPLE B. Steam from a steam boiler is blown into water in order to warm it. If 1 kilogramme of steam at five atmospheres, temperature 152°, is blown into 9 kilogrammes of ice-cold water, what will be the temperature of the 10 kilogrammes of warm water?

By Regnault's formula, if the steam were condensed to ice-cold water, the heat given out would be $1000 \times (606 \cdot 5 + 0 \cdot 305 \times 152)$, or 652,860 calories; but the steam really only condenses to t, the temperature of the mixture, so that actual heat given out is short by heat required to raise 1000 grammes of water from 0° to $t°$, that is, the heat given out in calories is

$$652{,}860 - 1000t.$$

Now the heat gained in calories by the 9 kilogrammes of water is $9 \times 1000 \times t$, or $9000t$. Hence

$$9000t = 652{,}860 - 1000t$$

from which we find $t = 65 \cdot 28$ degrees.

Or more simply, perhaps, the student may find by Regnault's formula the heat given out if the steam were condensed to water at 0°, viz, 652,860 calories, then find the temperature obtained on distributing this heat over the 10 kilogrammes of water, that is, divide by 10,000, which gives 65·286°.

Hygrometry. Terms used.

Dew-Point. The temperature at which the air if cooled would become saturated with the moisture it already contains.

Absolute Humidity. The actual mass of aqueous vapour in one cubic centimetre of air. (0·0005 gm., *e.g.*)

Relative Humidity. "The fraction expressing the ratio between the amount of vapour actually present in the air at a given temperature, and the greatest amount of vapour which it can contain at that temperature. The greatest amount representing complete saturation is generally reckoned equal to 100, and on this principle 50, 40, 30, &c., will denote that the air contains 50, 40, 30, &c., per cent. of the maximum amount which can be contained at that temperature."—Balfour Stewart, *Elementary Treatise on Heat*, 1888, page 152.

Hygrometric State is synonymous with relative humidity.

Hygrometric Quality is ratio between the weight of unit volume of air and that of the aqueous vapour present in it. —*Stewart.*

ABCD, fig. 45, represents section of a vessel fitted with a moveable weightless piston at *P*, the vessel contains a thermometer and a barometer. Suppose the space below the piston to be filled with dry air at the temperature indicated by the thermometer, and under the pressure indicated by the barometer. If the temperature = 10° and pressure = 700 mm., then knowing that 1 c.cm. of air at 0°C. and 760 mm. = 0·001293 gm., we find the mass of one cubic centimetre of air in the space *S* as

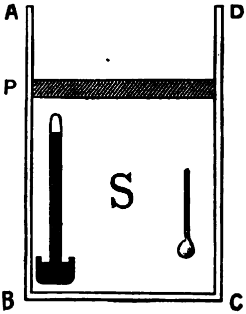

Fig. 45.

$$0\cdot001293 \times \frac{700}{760} \times \frac{273}{283} \text{ gm.}$$

If instead of dry air the space *S* contains vapour of water, then the mass of 1 c.cm. of this vapour of water would be calculated in a similar manner to that of the air, knowing that 1 c.cm. of water vapour at 760 mm. and 0°C. = 0·0008063 gm. Thus, mass of 1 c.cm. of vapour of water at 10°C. and 9·165 mm. pressure $= 0\cdot0008063 \times \frac{9\cdot165}{760} \times \frac{273}{283}$

In England the cubic foot is commonly used in hygrometry as a measure of capacity, and the grain, or pound, as measure of mass. When these units are adopted we require to know that a cubic foot of dry air at 0°C. and 30 inches barometer pressure is 566·89 grains, and a cubic foot of aqueous vapour at 0° and 30 inches = 353·46 grains.

Let us return to our first supposition, viz., that the space *S* contains dry air at 10° and 700 mm. pressure. Suppose drops of water to be now introduced into the space *S*, the water will slowly evaporate into the dry air, the piston will rise, but when the air is saturated the piston will no longer rise. The piston is then supported by two pressures, that of the dry air and that of the vapour of water, and these together = 700 mm. Now we know from Regnault's researches the pressure due to the vapour of water is 9·165 mm. (See table among Constants, *Tension of Aqueous Vapour.*) Hence that due to the air is 700 − 9·165, or 690·835, so that we have all data for calculating mass of the air and mass of the vapour of water.

Let the saturated air at 10° and 700 mm. pressure be warmed, say to 12°, the piston will rise, and we shall have now unsaturated air, but owing to the rise of the piston the pressure supported is the same as when it was saturated; hence, as before, 9·165 mm. of the pressure are due to the vapour of water, and 690·835 are due to the dry air.

From what has been said, it is clear, then, that if we know the whole pressure on the piston *P*, and the maximum pressure due to the vapour of water in the air of the space *S*, we can calculate the mass of 1 c.cm. of the water vapour in the space *S*. Now the maximum pressure can be found from the table of tensions of aqueous vapour when we know the temperature at which the air in the space *S* is saturated, that is, when we know the dew-point of the air in this space.

Finally, let the vessel and piston be removed so that the barometer and thermometer are simply immersed in the open air, and we then have the ordinary conditions of the meteorological observatory.

EXAMPLE C. The air temperature was 17°, the dew-point 13°, the barometer 750 mm. Find the absolute humidity of the air.

From table of tensions of aqueous vapour we find maximum tension of aqueous vapour at 13° = 11·162 mm.

Hence absolute humidity

$$=0·0008063 \times \frac{11·162}{760} \times \frac{273}{290} = 0·00001114,$$

that is, there is 0·00001114 gm. water vapour in each c.cm. of the air, or the absolute humidity is 0·00001114 gm. water per c.cm. of air.

EXAMPLE D. Find relative humidity of the air of last example.

Suppose that the space *S*, fig. 45, is filled with the air of Example C, that is, with air at 17° and a dew-point of 13°, the barometer being 750 mm., then the mass of 1 c.cm. of the water vapour in the air is 0·00001114 gm. We want now to know what will be the mass of 1 c.cm. of moisture in the air in *S* if it become saturated. Let water evaporate into the space *S*, the piston will rise, and when saturated the mass of 1 c.cm. of moisture will be (since maximum pressure at 17° = 14·421 mm.)

$$0·0008063 \times \frac{14·421}{760} \times \frac{273}{290}$$

But hygrometric state $= \dfrac{\text{Mass of 1 c.cm. of moisture at } 17^\circ\text{, dew-point } 13^\circ}{\text{Mass of 1 c.cm. of moisture at } 17^\circ\text{, dew-point } 17^\circ}$

that is,

$$\frac{0{\cdot}0008063 \times \dfrac{11{\cdot}162}{760} \times \dfrac{273}{290}}{0{\cdot}0008063 \times \dfrac{14{\cdot}421}{760} \times \dfrac{273}{290}} = \frac{11{\cdot}162}{14{\cdot}421}$$

or if 100 is maximum, the humidity $= \dfrac{11{\cdot}162}{14{\cdot}421} \times 100 = 77{\cdot}4$

from which it is obvious that the relative humidity can be obtained by dividing the pressure due to moisture at temperature of the dew-point at time of observation by the pressure due to saturated moisture at temperature of the air at the time of observation, and multiplying the quotient by 100.

I have illustrated this subject at some length, for even in works of repute there is confusion as to the proper method of determining the relative humidity or hygrometric state when the temperature of the air and of the dew-point are given.

Correction of a gas for moisture. Regnault's measurements of the maximum tension of aqueous vapour enable us to determine the mass of dry air or gas in a closed vessel if we take care to have the gas saturated with moisture. An example is given, but the student is referred to *Arithmetical Chemistry, Part II.* (Gas Analysis), for further information.

EXAMPLE E. The space *S*, fig. 46, is 20 cubic centimetres, and contains hydrogen saturated with moisture. The height of the mercury in the tube is 300 mm. A barometer near the apparatus reads 720 mm., and the temperature of the moist hydrogen is 12°. Find what would be the volume of the *dry* hydrogen if measured at 0° and 760 mm. pressure. (Tension of aqueous vapour at 12° = 10·457 mm.)

The 720 mm. pressure of the air on the cistern of the apparatus is balanced by three pressures—

(*a*) The column of mercury in the tube 300 mm.

(*b*) The pressure of saturated vapour in $S = 10{\cdot}457$ mm.

(*c*) The pressure of the dry hydrogen $x = 409{\cdot}543$.*

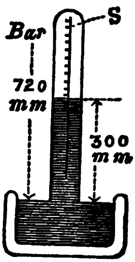

Fig. 46.

So that the actual pressure of the dry hydrogen is 409·543 mm., and its temperature is 12°. Knowing this, the volume (*V*) of the 20 c.cm. at 0° and 760 mm. is found in the usual way, thus

$$V = 20 \times \frac{409{\cdot}543}{760} \times \frac{273}{285} = 10{\cdot}2$$

---

* For 409·543 mm. are required in order that the three pressures shall make up 720 mm., or it may be put in this form: 720 mm. = 300 mm. + 10·457 mm. + x mm. Hence $x = 409{\cdot}543$.

EXERCISE X.

Of course in many of the following Exercises the student will use logarithms in preference to ordinary arithmetic.

1. What number of calories are required to evaporate 8 kilogrammes of water at 0° into steam at 150°?

2. What number of calories are given out in converting 10 grammes of steam at 80° into water at 0°?

3. What number of calories are given out in converting 10 grammes of steam at 80° into water at 80°?

4. Ten kilogrammes of steam at a pressure of 4 atmospheres and temperature 144° are blown into a vessel containing 100 kilogrammes of ice-cold water. What will be the temperature of the 110 kilogrammes of water?

5. The space S of fig. 46 is filled with *dry* air at a temperature of 12°, while the pressure on the weightless piston is equal to 720 mm. of mercury. Find the mass of 1 c.cm. of the air. What is the mass in grains of a cubic foot of this air when the pressure on the weightless piston is 28 inches of mercury?

6. The space S of fig. 46 is filled with air saturated with moisture; the temperature of the moist air is 20°, and the pressure on the piston 730 mm. Find the mass of dry air in 1 c.cm. of the moist air, and also the mass of vapour in 1 c.cm. of the moist air.

7. The saturated air of Example 2 is warmed to a temperature of 47°C., the pressure on the piston still remaining 730 mm. Find now the mass of 1 c.cm. of water vapour in the air, that is, find its absolute humidity.

8. At an observatory the air temperature was noted as 15°, the dew-point as 10°, and the barometer 720 mm. From these data, with help of table of tensions of aqueous vapour, find (i.) the absolute humidity of the air; (ii.) the relative humidity (per cent.).

9. The space S of fig. 45 contains air under the conditions of Example 8. Find to what extent the pressure on the piston must be increased in order that the air shall be just saturated.

NOTE.—If the air were cooled to 10° the air would be saturated; we must then apply such a pressure to the piston as would diminish the volume in the ratio of that which air has at 15° to that which it has at 10°, that is, we must increase the pressure in the ratio of $273+10 : 273+15$.

10. The space S of fig. 45 contains air under the conditions of Example 4. If the pressure on the piston be now doubled, what will be (i.) the absolute humidity of the air, (ii.) the relative humidity of the air?

11. A cubic mile of saturated air at 18° is cooled to a temperature of 10°. How many tons of water will be precipitated?

NOTE.—Find number of grammes precipitated in, say, a cubic metre of the air on cooling. Then as there are approximately $1{\cdot}07945 \times 10^{20}$ cubic metres in a cubic mile, multiply by this number (log = 20·03320), which gives weight in grammes. Now divide by $2{\cdot}264 \times 10^8$ (log = 8·55487) the number of grammes in a ton, and we have weight in tons.

In practice of course different units, cubic foot, and oz., *e.g.*, would be adopted, so as to avoid the conversion. With logarithms, however, the extra trouble in calculating is trifling.

ANSWERS.

1. 5,218,000 calories.
2. 6309 calories.
3. 5509 calories.
4. 59·13°.
5. 0·001173 gm.; 506·82 grains.
6. 0·001104 gm. (air); 0·0000168 gm. (vapour).
7. 0·00001538 gm.
8. 0·00000965 gm.; 72·17 per cent.
9. The pressure must be 732·6 mm., so that 12·6 mm. must be added.
10. 0·00001277 gm.; 100 per cent.
11. 2,938,400 approximately.

LESSON XI.

Conduction of Heat.

Conductivity. *The quantity of heat that flows, in unit time, through unit area of a plate of unit thickness, with one degree difference between the temperatures of its faces,* is defined as the *conductivity* of the plate.

An analogy may be made in the case of water to illustrate this definition.

Let *abcd*, fig. 47, represent a pipe just one square centimetre in section and one centimetre long, connecting two tanks, the level of the water in the left-hand tank being always maintained one centimetre above the level of the water in the right-hand tank. Then the "conductivity" of the pipe for water will be expressed by the number of cubic centimetres of water that flow through the pipe *abcd* in one second. If gratings are introduced into the pipe to check the flow, the number of cubic centimetres passing per second will vary according to the size and number of the gratings.

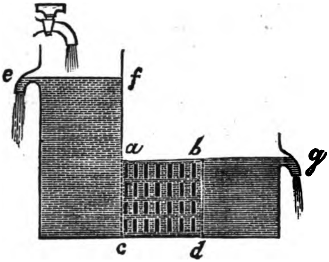

Fig. 47.

Fig. 48 represents an imaginary experiment in conduction of heat. *B* is a block of metal whose conductivity is being tested. The block is a cube of one centimetre along one edge, and against two of its opposite faces are the vessels *A* and *C*, each containing one cubic centimetre of water. The water in *A* is always maintained by means of a lamp at a temperature 1° above that in *C*. Now under these circumstances, since *C* contains one cubic centimetre of water, each degree rise in temperature indicates one calorie transferred from *A* to *C* through the

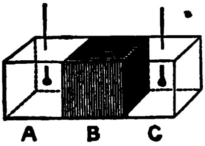

Fig. 48.

conductor B. Hence were it found that in one second the temperature of C had risen one degree, the conductivity of the material B would be 1; if the temperature of C rise 0·5 degree in one second, the conductivity is one-half, and so on.

Suppose that the unit block of fig. 48 transmits q calories per second, and we want to know what quantity (Q) some other block of the *same* material will transmit under similar circumstances, when we know its length (l), its area (A), and the number of seconds (t) during which the steady flow of heat has taken place from a temperature v_2 at one end of the bar to a temperature of v_1 at the other end; then if we keep in view that

Q will be greater as A is greater
" " " " t "
" " " " $v_2 - v_1$ is greater
" " less " l is greater

it follows that

$$Q = q\,\frac{A \times t \times (v_2 - v_1)}{l}, \text{ or}$$

$$q, \text{ that is, conductivity of the block} = \frac{Ql}{At\,(v_2 - v_1)}$$

which is the formula for conductivity usually given.

To determine absolute conductivity, an experiment analagous to that of fig. 48 may be made, but the relative conductivity of metals can be determined by taking bars of equal size of the metals, heating one end, and noting by thermometers at equal distances the temperatures along the bars, and plotting curves to indicate the fall of temperature. If we suppose a number of bars of different metals of uniform cross section, all having one end heated to the same temperature (by a bath of molten lead, for example), then the temperature curves obtained by a series of thermometers placed at equal distances along the bars will be steeper as the conductivities are less; in fact, if we note the tangents of the angles of the slopes at points where the same temperature occurs, we have a series of numbers which are inversely as the conductivities of the metals.

Having determined the absolute conductivity for one metal, and the relative conductivities for others, the absolute conductivities of metals can be given. The student should read Chapter VII. of *An Elementary Treatise on Heat*, by Balfour Stewart (Frowde), for more detailed information. It may, however, be pointed out here that the transmission of heat along a conductor is somewhat complex. When a long bar is first heated at one end it communicates heat to the particles near it; these when heated communicate it to the next row of particles, and so on. Now, irrespective of conduction, radiation, or convection, some time may be consumed in propagating this heat, owing to the material of the bar having a high specific heat. Again, when the surface of the bar is heated it communicates heat to the walls of the room by radiation, it also warms the air of the room, and convection currents are set up; it is that heat which finally reaches the end of the bar that can communicate energy to the water *C* in fig. 48, and it is this heat only which is transmitted by conduction. Hence, to make the experiment represented in fig. 48, the block *B* must be coated with a non-conductor, to prevent radiation and convection, and before measurements are made sufficient time must be allowed for the temperature of the bar through its whole length to be steady.

Principal Forbes applied the method of flow of heat along a bar to investigate the question of alteration of conductivity with temperature, and he found generally that the conductivity of iron diminished as the temperature was raised. Hence heat resembles electricity. By an expedient, which will be understood by reference to Stewart's *Heat*, the absolute conductivity of iron was also found.

Reverting to the analogy in the case of water (fig. 47), it is evident that when the water first begins to flow through the pipe time will be consumed in passing the several gratings, and the flow will be quicker as the gratings are fewer; thus, a pipe with many gratings will correspond to a bad conductor, while one with few gratings will represent a good conductor. If we suppose the pipe to have side pouches, into which water can flow, time will be consumed in filling these pouches, and we have the analogue of specific heat. Again, if we suppose the pipe to be leaky along its whole length, the escape of water by the leaks will represent the loss by radiation and conduction; finally, pressure on the sides of the pipe will correspond to temperature. In the language of the molecular theory of heat, the transmission of heat along the bar consists in a communication of motion from molecule to molecule until the end of the bar is reached. Time is required for this communication. If the molecules have to be turned about their centres, or any internal motion to be given, time is consumed in giving this motion (specific heat), the moving molecules communicate motion to the ether at the surface of the bar (radiation), and also to the air in contact with the sides of the bar (expansion of the air, and ultimately convection).

EXAMPLE A. The exposed surface of a steam pipe containing steam at 110° measures 5 square metres, the outer surface has a temperature of 109°, and the thickness of the pipe is 0·5 cm. How many calories of heat per hour are lost by conduction from this exposed surface? (Take conductivity of iron as 0·15 in. C.G.S. units.)

By formula developed at *p*, we find that

$$Q = \frac{0{\cdot}15 \times (5 \times 10^4) \times 60 \times 60 \times (110 - 109)}{0{\cdot}5}$$
$$= 54 \times 10^6$$

EXERCISE XI.

1. A bar of metal 10 centimetres long and 2 square cm. area, protected with a non-conductor, so that there was little loss by radiation or convection, transmitted 1 calorie per second, with a difference of 10 degrees between its ends. What is approximately its conductivity?

2. The conductivity of iron being taken as 0·2 C.G.S. units, find what weight of ice per hour could be just melted over every square decimetre of the surface of a large iron plate, the thickness of which is 5 centimetres, and the temperature of the under side maintained at 200°C.

3. One side of a room is composed of a brick wall 14 inches thick (conductivity 0·00147), in which are glass plates ¼ inch thick (conductivity of glass 0·00234). If the temperature of the inside of the room is to be maintained constant, then, as regards conduction only, how many times more heat per square foot would have to fall upon the glass than upon the brick?

ANSWERS.

1. 0·5.
2. 360 kilogrammes.
3. 89·12 nearly.

LESSON XII.

Mechanical Equivalent of Heat. Ratio of Specific Heat at Constant Pressure to Specific Heat at Constant Volume.

Mechanical Equivalent of Heat. Methods by which Joule determined the mechanical equivalent of heat are given in detail in "The Scientific Papers of James Prescott Joule," Physical Society of London, Taylor and Francis, 1884. An outline of each method is given below, reference to the pages of the work following each description.

Method 1. Fluid Friction The drum-shaped copper vessel *ll*, fig. 1, plate 8 (nine inches high and nine inches diameter), has four perforated projections from the sides towards the centre, as shown at *bbbb* in the cross section of the vessel, fig. 2, and again at *b* in the vertical section in fig. 3 Within this vessel an eight-rayed fan, with key-like leaves or paddles, could be turned by two weights attached to strings wound round a wooden cylinder on the axle of the fan. The arrangement of the whole is represented in fig. 1, excepting that a second pulley and weight, corresponding to that shown, is left out. Plate 8.

In making the experiment, a known weight of water is placed in the box *ll*, and its temperature recorded by immersing an extremely sensitive thermometer in the aperture *t*. The thermometer is now removed, and the weights *e* allowed to fall, when the eight-rayed paddle as it works through the water warms it slightly. The weights are wound up by turning the handle *m* (which can be done without disturbing the paddle), and they are allowed to fall a second time; the operation is repeated many times, after which the thermometer is replaced, and the temperature again noted. The temperature of the room is carefully observed that any accidental heating or cooling by radiation from surrounding objects may be corrected. Knowing weight of water in the copper vessel, thermal value of the box and paddles in terms of water, the rise of temperature, and the actual mechanical work done by the movement of the weights, the value of the mechanical equivalent was known. (*Joule*, page 298.)

Method 2. By work done in compressing air. It was first shown by the apparatus represented in fig. 49 that if the receiver *R* was filled with compressed air, the temperature of which had sunk to that of the room, and communication made suddenly with the exhausted vessel *E*, that no rise of temperature took place in the water by which the two vessels were surrounded. This important experiment shows that when air expands *without doing work*, if such be possible, on the whole no heat is given out or absorbed, and conversely it may be inferred that in compressing air (removing the heat as fast as formed into water, so that the air does not rise much in temperature) the heat produced is entirely that due to mechanical work done in compressing the air.

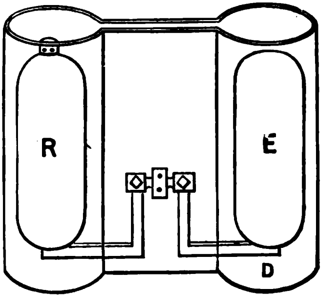

Fig. 49.

The apparatus used after this fact had been established is represented in fig. 50. The copper receiver *R* is surrounded by a known weight of water. The pump *C* is used to compress the air, which is drawn through the drying vessel *G* and tube *W*.

The temperature of the water having been noted by the thermometer t_1, and of the air of the room by the thermometer t_2 (the thermometers were particularly sensitive), 300 strokes were made with the pump so as to condense the air about 22 times; after this the water was stirred, and its temperature noted. Now the

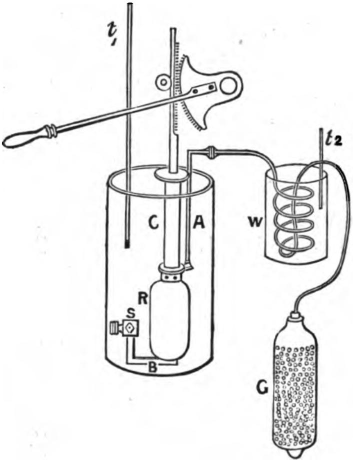

Fig. 50.

rise of temperature in the water was due to two causes: first, the friction of the piston of the pump; and, second, the conversion into heat of the work done in compressing the air. The heat due to the friction of the pump was determined by a separate working of the pump with the air pipe *A* closed, and a vacuum under the piston, as also with varying pressures of air under the piston. Knowing the allowance to be made for friction of the pump, it only remained to determine the mechanical work done in compressing the air when the direct relation between mechanical energy and rise in temperature of a mass of water was obtained. (*Joule*, 172-189.) The mode of calculating the energy expended was that of the indicator diagram. (See Reference Chapter A.)

Method 3. Magnetic Brake. An electro-magnet was sealed up in a glass tube filled with water (terminal wires passing through the tube to a mercury commutator), and this magnet was rotated in a magnetic field, produced by a powerful, fixed electro-magnet. Knowing the mechanical work expended in rotating the electro-magnet against the resistance of the powerful field of force, and also the rise in temperature of the glass, water, iron, and copper, which formed the rotating piece, the mechanical equivalent was determined. (*Joule*, 123-159.)

Method 4. By heat from voltaic currents. By means of a current-meter the value of a voltaic current could be determined in absolute measure, that is, it could be ascertained with any current used for a time (40 minutes suppose) to what mechanical energy this was equal. Knowing the mechanical energy expended in maintaining the current, it was now only necessary to pass the current through a known resistance and communicate the heat liberated to a mass of water when the requisite data were supplied. (*Joule*, 542-557.)

Joule established by his experiments that if a weight of one pound were lifted to a height of 1390 feet (more accurately 1389·6), and its potential energy converted into heat, the heat would be sufficient to raise one pound of water one degree centigrade. The experiment was made in Manchester, and, strictly speaking, the value of gravity at Manchester should be included in the statement to give a correct idea of the relation; for had the experiment been made at the equator, a somewhat higher number would have been obtained, while had the experiment been made at the North Pole, it would have been a somewhat lower number.

Since a mass of *one pound* falling 1389·6 feet at Manchester will raise *one pound* of water at 1°C., a mass of *one gramme* falling 1389·6 feet at the same place will raise *one gramme* of water 1°C.; and as 1389·6 feet are equal to 42,355 centimetres, we arrive at the statement that the energy of a mass of one gramme which has fallen through 42,355 centimetres at Manchester is equivalent to one calorie. If we wish to express the value of the mechanical equivalent in ergs, we must multiply by the value of gravity at Manchester (981·3 dynes), which gives the unwieldly number 41,462,961·5. Some later determinations of the mechanical equivalent of heat give 42 million ergs, or $4·2 \times 10^7$, and this is the number adopted by Everett in *Physical Constants*. To sum up, we may say that the mechanical equivalent of heat, or *Joule*, is

$a = 1390$ foot-pounds (772 for 1°F.)
$b = 44,758$ foot-poundals ($g = 32·2$).
$c = 42,355$ gramme-centimetres.
$d = 4·2 \times 10^7$ ergs.

Any of the above expressions may be represented by the symbol *J*. So we have

1 calorie (c) = 1 Joule (J) = 42,355 gramme-centimetres = $4·2 \times 10^7$ ergs.

1 lb. degree C. = 1 Joule (J) = 1390 foot-pounds = 44,758 foot-poundals if $g = 32·2$.

In applying Joule's equivalent to questions involving the motion of a mass, the kinetic energy of the moving mass must first be found. This, as explained in *Arithmetical Physics, Part II*B, is the product of half the mass into the square of the velocity. Thus, a cannon ball of 20 kilogrammes mass, moving with a velocity of 36,000 centimetres per second, will have a kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}(20 \times 1000) \times (36,000)^2$, or $1·296 \times 10^{13}$ ergs, and this, since $J = 4·2 \times 10^7$ ergs, will equal $1·296 \times 10^{13}/(4·2 \times 10^7)$, or $3·0857 \times 10^5$ Joules; and this may be taken as meaning that if the ball were arrested, and its energy applied to warm a mass of water, it would raise $3·0857 \times 10^5$ grammes of water one degree centigrade.

EXAMPLE A. In order to raise 10 gallons of water through 50°C., how many foot-pounds of mechanical work must be expended?

Ten gallons of water = 100 lb., and as 1390 foot-pounds of work will raise 1 lb. of water 1°C., it will require 1390×50, or 69,500 foot-pounds to raise 1 lb. of water 50°, and 100 times this, or 6,950,000 foot-pounds, to raise 100 lb. of water 50°C.

EXAMPLE B. A gallon of water at 100°C. is converted into steam at 100°C., and again condensed to water at 100°C. Supposing that one-half the energy given out in condensing the steam is used to lift a weight of one ton from the bottom of a coal pit, to what height will the weight be lifted? (Take latent heat of steam as 537.)

Assuming the gallon of water as 10 lb., the heat given out by the condensation is 10×537, or 5370 pound-degrees, and each pound-degree is equivalent to 1390 foot-pounds; hence foot-pounds of energy given out by the steam in condensing is 5370×1390, or 7,464.300, of which one-half, or 3,732,150, is available. Now 1 ton is 2240 lbs., so height to which it will rise is 3,732,150/2240, or 166·6 feet nearly.

EXAMPLE C. One gramme of charcoal when burnt will raise 8080 grammes of water one degree centigrade, that is, will produce 8080 calories. If a kilogramme of charcoal is burnt, how many ergs of work will be liberated?

Number of calories liberated by burning the 1000 grammes of charcoal $= 8080 \times 1000$, or 8 080,000, and this in ergs $= 8{,}080{,}000 \times 4{\cdot}2 \times 10^7 = 3{\cdot}3936 \times 10^{14}$.

EXAMPLE D. The energy given out by the burning of a kilogramme of charcoal is used to raise a weight of 1000 kilogrammes at a place where $g = 1000$. Find to what height the weight would be raised supposing all the energy available.

From Example C number of ergs given out by the burning of a kilogramme of charcoal $= 3{\cdot}3936 \times 10^{14}$. Number of ergs required to raise 1000 kilogrammes, or 1,000,000 grammes, *one centimetre* at a place where $g = 1000$ is $1{,}000{,}000 \times 1000$, or 10^9. Hence height to which the mass will be raised will be $3{\cdot}3936 \times 10^{14}/10^9 = 3{\cdot}3936 \times 10^5$, or 339,360 centimetres.

NOTE.—The above examples contain many varieties of illustrations of the use of the Joule. As regards the particular value to be used, it will, I believe, be found that in chemical problems, as in *Arithmetical Chemistry II., Lesson XIII.*, the C.G.S. system of units is most convenient ($J = 4{\cdot}2 \times 10^7$), but in mechanical problems the English value ($J = 1390$ foot-pounds) is simpler. The student should, however, be familiar with both.

Specific Heat at Constant Pressure, and Specific Heat at Constant Volume. As already stated, Joule showed that when a gas (actually air) expands *without doing work* its temperature remains the same; if, however, the air in expanding does work, then it is cooled, and the amount of cooling measures the work done.

Consider a tube, as in fig. 51, one square centimetre in area, of indefinite length, and having a weightless air-tight piston at W, 273 centimetres from the bottom. On the weight-

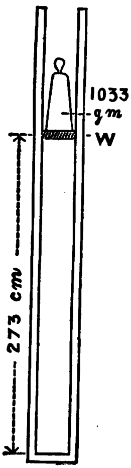

Fig. 51.

less piston let a weight of 1033 grammes be placed. Suppose the tube below the piston to contain air at 0°C. and 76 cm. pressure, and the whole arrangement placed in a vacuum, the piston will then remain at the mark 273. Or since 1033 grammes per sq. cm. is simply the atmospheric pressure, we may keep the weightless piston in its place; remove the weight, and let the atmospheric pressure alone act. Suppose now the temperature of the air to be raised 1°C., the piston will rise to 1 centimetre, and work equal to lifting 1033 grammes 1 centimetre high against gravity will have been performed, and we can easily find what energy has been expended in the operation. For if the piston and weight had been raised 42,355 centimetres, there would have been 1033 Joules performed, and, of course, for one centimetre there would have been 1033/42355, or 0·0244 Joule, that is, the work is equal to raising 1 gramme of water 0°·0244 C., or it is 0·0244 calorie.

Now, from Regnault's experiments on the specific heat of air under pressure, it is known that to heat the 273 cubic centimetres of air (weighing 0·3522 gm.) in the tube at the same time that the 1033 grammes is lifted (that is, allow the air to expand under ordinary atmospheric pressure) requires 0 0835 calorie; hence of the total heat applied to the air (0·0835 – 0·0244), or 0·0591 calorie, is employed in raising its temperature 1°C., and 0·0244 calorie is consumed in expanding the air against atmospheric pressure, that is, in raising the weight, making together the total 0·0835, as derived from Regnault. Hence we learn that

Heat required to raise 273 c.cm. of air 1° without doing work by its expansion (*constant volume*) } = 0·0591 calorie,

Heat required to raise 273 c.cm. of air 1° with expansion against atmospheric pressure (*constant pressure*) } = 0·0835 calorie,

$$\text{or } \frac{\text{specific heat of air at con. pressure}}{\text{specific heat of air at con. volume}} = k = \frac{0{\cdot}0835}{0{\cdot}0591} = 1{\cdot}412.$$

This number, 1·412, is then the ratio of specific heat at constant pressure to specific heat at constant volume.

The above method of finding the ratio of specific heats is an extremely important one. Two other methods may be referred to:—

Method of Clément and Désormes. Suppose a gramme of air at 0°C. confined in a vessel with moveable piston with a weight upon it corresponding to atmospheric pressure. Let the air be now warmed to a temperature of 273°, when we know from Regnault's experiments that the heat absorbed is $273 \times 0{\cdot}2375$, or 65 calories nearly, while the volume of the air has doubled. Suppose now the vessel and piston impervious to heat, and that the piston be suddenly pressed down so as to restore the air to its original volume, we have now the 65 calories engaged in raising the temperature of the original volume of air, and it is evident that if the temperature is unaffected, the specific heat is the same, whether the air remains at its original volume, or whether it be allowed to expand. As a matter of fact, however, the air rises to a temperature of about 388°; we know then that while 65 calories will raise a gramme of air to a temperature of 273° if pressure be constant, the same 65 calories will raise it to a temperature of 388° if its volume is constant; so the ratio of the two specific heats is 388/273, or 1·4. Clément and Désormes ascertained the change of temperature produced when the volume of a gas was suddenly altered, and thus obtained the ratio by the principle above given. There were, however, great difficulties in the way, and their results now only form a part of the history of the subject.

Method by velocity of sound in a gas. We have seen in Lesson I., page 6, that the ratio we are discussing enters as a factor in the expression for velocity of sound in a gas. Hence, finding the actual velocity of sound in various gases, we may find the required ratio. This method was applied by Dulong (Watt's *Dictionary of Chemistry, III.*, 1865, page 42) to several gases. (See *Tables and Constants* near end of book.)

EXAMPLE E. The velocity of sound in olefiant gas was found to be 31,590 centimetres per second at 0°C., and 76 cm. mercurial pressure $g=981$, its density at the same temperature and pressure $=0{\cdot}00126$ gm per c.cm. Required the ratio of specific heat at constant pressure to specific heat at constant volume.

From Lesson I., page 8, we have

$v=\sqrt{\frac{P}{d}\,.\,k}$ if v = velocity, P = pressure gas supports, d = its density, and k = required ratio, or

$$k=v^2\times\frac{d}{P}=(31{,}590)^2\times\frac{0{\cdot}00126}{76\times13{\cdot}6\times981}=1{\cdot}24.$$

It will be noticed that k is here a much lower ratio than with air (1·41); this means, of course, that the specific heat at constant volume of olefiant gas is greater than air. This subject is entered into fully in Lesson XV.

The method of Example E has been applied by Kundt (*Pogg. Ann*, 1876, page 353) to determine the ratio of specific heats in the case of mercury vapour. It is shown in this investigation that the ratio is 1·66. This implies that the mercury molecule is extremely simple in constitution, and supports the views of chemists in this respect.

EXERCISE XII.

1. A cubic foot of water falls from a temperature of 20°C. to 10°C, Find (*a*) number of pound-degrees this is equal to; (*b*) number of foot-pounds of energy this is equal to; (*c*) number of foot-poundals of energy this is equal to. (One cubic foot of water = 62·5 lb., g = 32 poundals.)

2. A cannon ball having a mass of 50 lb. is projected from a cannon with a velocity of 1000 feet per second. (*a*) Find its kinetic energy, that is, number of foot-poundals of work that it would do if stopped; (*b*) supposing its energy used to raise the temperature of 10 lb. of water to 0°C., to what temperature will the water be raised?

3. As we have seen, page 105, the Joule = 42,355 gramme-centimetres where gravity is as at Manchester 981·3 dynes. What would be the value of the Joule in gramme-centimetres at a place where gravity was double this, or 1962·6 dynes?

4. A cannon ball 25 kilogrammes mass is projected from a cannon with a velocity of 40,000 centimetres per second. Find (*a*) its kinetic energy, or the number of ergs of work that must be done to put it in motion; (*b*) what this is in calories.

5. A railway train has a mass of 108,000 kilogrammes, and is moving at the rate of 60,000 metres an hour, and hence its kinetic energy in ergs is $1{\cdot}5\times10^{14}$. To how many calories is this equal?

6. A train having a mass of 40 tons runs up an incline of 1 in 30 at a speed of 20 miles per hour. Find (*a*) the kinetic energy of the train in foot-poundals; (*b*) the work in foot-poundals per minute consumed in lifting the train up the incline; (*c*) if the friction of the wheels and other parts were 1/200ths of the load, what work in foot-pounds per minute would be done in overcoming this friction?

NOTE.—The work of friction is measured in foot-pounds by multiplying the friction of the load in pounds into the feet through which the load is carried per unit of time.

7. From what height in metres must a block of ice be dropped so that when stopped, its energy, assumed as given up entirely to the ice, may be just sufficient to melt it? (Latent heat of water = 80.)

8. Water at the top of a waterfall has a velocity of one metre per second, it then falls into the basin below, so that just before reaching the basin it has a total velocity of 50 metres per second. To what energy in ergs is one gramme of the moving water equal, and by what fraction of a degree centigrade would the temperature of the water be raised by the arrest of its motion?

9. A kilogramme of iron at 0°C. falls from a height of 42,355 centimetres, and is then stopped. If the heat produced appears entirely in the iron, to what temperature will it be raised? (Specific heat of iron being taken at 0·1.)

10. A horse-power is considered as that amount of energy which will in one minute raise 33,000 lbs. through a height of 1 foot against gravity. To how many pound-unit degrees per second is this energy equal?

11. Express a horse-power in gramme-centimetres, taking the following approximate numbers:—1 lb. as 453 grammes, and 1 foot as 30 centimetres. Express it also in ergs, where $g = 1000$.

12. A cubic metre of air at 0°C. is warmed until it attains a temperature of 273°C., and has doubled its volume against an atmospheric pressure of 1033 grammes per sq. cm., $g = 981$. Find how many ergs of work have been done in the expansion of the air against the pressure.

13. The specific heat of air under constant pressure being taken as 0·24, find the whole energy in ergs consumed in warming and expanding the cubic metre of air under the conditions of Question 12. (One cubic metre of air at 0°C. and 1033 gm. per sq. cm. pressure being taken as 1293 grammes.)

14. From the data of Questions 12 and 13, find (i.) how many times the specific heat of air at constant pressure is greater than the specific heat of air at constant volume; (ii.) the actual specific heat of air at constant volume.

15. A box containing just one cubic metre of air at 0° and 76 cm. pressure is hermetically sealed, and the confined air then warmed to 200°C. What number of calories are required to effect this change of temperature?

16. A cylinder is provided with an air-tight piston. The cylinder contains just one cubic metre of air at 0°C. and 76 cm. pressure. The air in the cylinder is warmed, and as it is warmed the piston is gradually lifted by some mechanical contrivance, so that the air as it expands does no work. What number of calories will be required to warm the air under these conditions to a temperature of 200°?

17. A kilogramme of hydrogen is burnt to steam at 100°C., and supposing the energy liberated could be used to raise a mass of 1000 kilogrammes against gravity where $g = 980$, find the height to which this mass would be raised.

NOTE.—The energy given out by one gramme of hydrogen, burnt according to the conditions of the question, may be taken as 29,000 calories nearly. Take $J = 4{\cdot}2 \times 10^7$ ergs.

18. The velocity of sound in ether vapour is approximately 18,000 cm. per second, while its density is 0·0033 gm. per c.cm, at 0°C., and under a pressure of 76 cm. of mercury, $g = 981$. What is the ratio of specific heat at constant pressure to specific heat at constant volume in ether vapour?

ANSWERS.

1. (*a*) 625 pound-degrees; (*b*) 868,750 foot-pounds; (*c*) 27,800,000 foot-poundals.
2. (*a*) 25×10^6; (*b*) 56·2°.
3. 21,177·5 gm. cm.
4. 2×10^{18}; $4{\cdot}76 \times 10^5$.
5. $3{\cdot}57 \times 10^6$.
6. (*a*) $1{,}141{,}133{\cdot}\dot{3}$ foot-poundals; (*b*) 25,231,600 foot-poundals; 78,848 foot-pounds.
7. 33,884 metres.
8. $1{\cdot}25 \times 10^7$ ergs; 0·3° nearly.
9. 10°.
10. 0·4 nearly.
11. $4{\cdot}4847 \times 10^8$; $4{\cdot}4847 \times 10^{11}$.
12. $1{\cdot}013373 \times 10^{12}$.
13. $0{\cdot}24 \times 1293 \times 273 \times 4{\cdot}2 \times 10^7 = 3{\cdot}5581 \times 10^{12}$.
14. (i.) 1·404; (ii.) $0{\cdot}24 \times \frac{1}{1{\cdot}404}$, or 0·1709.
15. $0{\cdot}1709 \times 1293 \times 200$, or 44,203 calories.
16. The same as in 15.
17. $124{\cdot}387 \times 10^7$ metres.
18. 1·054.

LESSON XIII.

Work Done in Compressing Air.

Isothermal Compression and Adiabatic Compression. Two cases present themselves in compressing air. First, when the rise in temperature produced by compression is prevented by carrying away the heat as fast as it is produced. With these conditions we have *isothermal compression.*

Plate 8. Suppose the air to be compressed contained in a cylinder with an air-tight piston at *b*, as represented in fig. 2, plate 8. Draw the two lines *OV* line of volumes, and *OP* line of pressures. Mark the point *a*, making *ab* equal to p_1 the pressure of the air in the cylinder before compression, mark other points corresponding to different pressures until we come to p_2, at which put *c* to indicate the extremity of the line indicating the pressure p_2. The curve *ca* is portion of a rectangular hyperbola, and the area *abcd* represents the work done in moving the piston from *b* to *d*, while the volume changes from v_1 to v_2. By supposed conditions the temperature remains constant, so the work *W* done in this isothermal compression is expressed by the area of *abcd*, that is (see Reference Chapter A),

$$W=p_1v_1 \times 2{\cdot}302585 \times \log \frac{v_1}{v_2}$$

Thus, if we take as units the pound and the foot, then to find the work done in compressing 1·7106 cubic foot of air under a pressure of 2136 lb. per square foot into the space of 0·0790 cubic foot, the temperature remaining constant, we have

$$\begin{aligned} W &= 2136 \times 1{\cdot}7106 \times 2{\cdot}302585 \times \log \frac{1{\cdot}7106}{0{\cdot}0790} \\ &= 2136 \times 1{\cdot}7106 \times 2{\cdot}302585 \times 1{\cdot}33552 \\ &= 11{,}237 \text{ foot-pounds.} \end{aligned}$$

Of course if other units are adopted, the result will be obtained in other units.

The result may also be obtained, though less accurately, by construction. Thus in fig. 2, plate 8, consider the piston as one square foot in area, and suppose the vertical lines are drawn at successive foot-lengths of the cylinder. Further, let the pressure per square foot, represented by successive distances on the line *OP*, be each equal to 10 lb. per square inch, or 1440 lb. per square foot. Then each square in the indicator diagram *abcd* represents 1440 foot-pounds of work; and to find the work done in compressing 12 cubic feet of air under a present pressure of 2160 lb. per square foot to a volume of 5 cubic feet, we set off the successive pressures as the volume diminishes, obtaining the figure *abcd*; on now counting the squares included, we find there are about 15·5 squares, and as each = 1440 foot-pounds of work, we have 15·5 × 1440, or 22,320 foot-pounds as the work done. By calculation the work is 22,690, the discrepancy being due (*a*) to defects in drawing, and (*b*) to difficulty in estimating the area of the part squares.

Suppose the piston and cylinder of fig. 2, plate 8, impervious to heat, then as the piston is forced down the temperature of the air rises, the resistance is greater, and by the time the piston reaches *d* the pressure p_2 will be much higher than before = *c′d* suppose. If the successive pressures are plotted out, the curve *c′a* will be formed, a curve known as the adiabatic curve, in which the relation $p_1 v_1 = p_2 v_2$ no longer holds, but in which

$$p_1 v_1{}^{\gamma} = p_2 v_2{}^{\gamma} \qquad (a)$$

where γ is the ratio of specific heat at constant pressure to specific heat at constant volume This ratio for air and most gases is 1·408; hence the equation for air is

$$p_1 v_1{}^{1\cdot 408} = p_2 v_2{}^{1\cdot 408}, \text{ or } p_2 = p_1 \left(\frac{v_1}{v_2}\right)^{1\cdot 408} \qquad (a')$$

From this equation we can calculate one of the quantities when the other three are given. As in the isothermal curve, so in the adiabatic one, the area *abc′d* of fig. 2, plate 8, represents the work done. This area may be shown to be

$$\frac{p_2 v_2 - p_1 v_1}{\gamma - 1}, \text{ or for air } \frac{p_2 v_2 - p_1 v_1}{0\cdot 408}\text{; hence}$$

$$W = \frac{p_2 v_2 - p_1 v_1}{0\cdot 408} \qquad (b)$$

so that knowing the original volume, the final volume, and the final pressure, the work of compression may be calculated.

From *Arithmetical Physics, Part I*A, page 32, we learn that when volume is constant the pressure varies as the absolute temperature, that is, as $273+t$ where t is temperature, as indicated by a centigrade thermometer. Also if the pressure is constant the volume varies as the absolute temperature. Hence if v_1 is volume of a gas at temperature t_1°C., and under pressure p_1, and then by any alteration its temperature is changed to t_2, its volume to v_2, and its pressure to p_2, we shall have the relation

$$\frac{273+t_2}{273+t_1}=\frac{v_2}{v_1}\cdot\frac{p_2}{p_1} \qquad (c)$$

This relates to isothermal compression, but in adiabatic compression we have from a'

$$\frac{p_1}{p_2}=\left(\frac{v_2}{v_1}\right)^{1\cdot408} \quad \text{(i.)} \qquad \text{and } \frac{v_1}{v_2}=\left(\frac{p_2}{p_1}\right)^{\frac{1}{1\cdot408}} \quad \text{(ii.)}$$

From (c) and (i.)

$$\frac{273+t_2}{273+t_1}=\frac{v_2}{v_1}\cdot\left(\frac{v_1}{v_2}\right)^{1\cdot408}=\left(\frac{v_1}{v_2}\right)^{0\cdot408} \qquad (d)$$

or from (c) and (ii.)

$$\frac{273+t_2}{273+t_1}=\left(\frac{p_1}{p_2}\right)^{\frac{1}{1\cdot408}}\cdot\frac{p_2}{p_1}=\left(\frac{p_2}{p_1}\right)^{\frac{0\cdot408}{1\cdot408}}=\left(\frac{p_2}{p_1}\right)^{0\cdot293}$$

$$\text{or } 273+t_2=(273+t_1)\left(\frac{p_2}{p_1}\right)^{0\cdot293}$$

$$\text{since } \frac{0\cdot408}{1\cdot408}=0\cdot293 \text{ nearly.}$$

The above transformations will be puzzling to the student not familiar with the theory of indices. (See Reference Chapter A for an explanation.)

EXAMPLE A. Required the work done in compressing 4 cubic feet of air at 14·7 lb. per square inch into a space of 1 cubic foot, supposing that no heat escape (adiabatic compression).

If in the adiabatic curve $c'a$ we take p_1 as 14·7 × 144, or 2116·8 lb. per square foot, v_1 as 4, v_2 as 1, we have all the quantities but p_2 with which to calculate the area $c'abd$ by formula b. We first find p_2.

$$\text{Now from } (a')\ p_2=p_1\left(\frac{v_1}{v_2}\right)^{1\cdot408}$$

$$=2116\cdot8\times4^{1\cdot408}\ *$$

$$=2116\cdot8\times7\cdot042, \text{ or } 14{,}906\cdot5 \text{ lb. per square foot.}$$

Having now the value of p_2, we have the area of $c'abd$, and, consequently, the work done as

$$\frac{14{,}906\cdot5\times1-2116\cdot8\times4}{0\cdot408} \quad \text{(See equation } b.\text{)}$$

$$=\frac{14{,}906\cdot5-8467\cdot2}{0\cdot408}$$

$$=\frac{6439\cdot3}{0\cdot408}=15{,}784 \text{ foot-pounds.}$$

EXAMPLE B. 20 cubic feet of air at 15°C. and 14·7 lbs. pressure are suddenly reduced to 5 cubic feet without loss of heat. Required the temperature of the gas.

By formula d, page 114, if $273+t_2=\tau_2$,

$$\frac{273+15}{\tau_2}=\left(\frac{5}{20}\right)^{0\cdot408}, \text{ or } \tau_2=288\times\left(\frac{20}{5}\right)^{0\cdot408}$$

$$=288\times4^{0\cdot408}=288\times1\cdot76=506\cdot9 \text{ nearly.}$$

As the absolute temperature $\tau=506\cdot9$, the temperature $=506\cdot9-273=233\cdot9$°C.

EXERCISE XIII.

1. Draw two lines at right angles to each other, mark off a point on the horizontal line 273 mm., or if more convenient, 273 eighths of an inch from their intersection, that is, make OV in fig. 2, plate 8, $=273$. On the vertical line mark off equal parts each equal to 2 mm., or ¼ in. Now plot out a curve illustrating *isothermal compression.*

2. Again, with the same or similar lines plot out a curve illustrating the adiabatic compression of 273 volumes of air, taking for simplicity that

$$p_1v_1^{\,2}=p_2v_2^{\,2}$$

3. When the curves of problems 1 and 2 are sufficiently complete, show *by construction* that the area corresponding to *abcd* of fig. 2, plate 8, is equal to $p_1v_1\times2\cdot302585\times\log.\frac{v_1}{v_2}$, and that in the adiabatic curve it is equal to $\frac{p_2v_2-p_1v_1}{2-1}$.

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[*] To find this value find the logarithm of 4, viz., ·60206. Multiply this into 1·408, which gives ·84770. Hence, in logarithms, $\log p_2=\log 2116\cdot8+\log 4^{1\cdot408}=\log 2116\cdot8+0\cdot84770=3\cdot32568+0\cdot84770=4\cdot17338$, so that $\log p_2=4\cdot17338$, and $p_2=14{,}907$ nearly.

4. Find the work in foot-pounds requisite to compress 4 cubic feet of dry air at a pressure of 14·7 lb. per square inch = 2116·8 lb. per square foot into a space of 2 cubic feet, first by isothermal compression, second by adiabatic compression.

5. A cylinder of 1 square inch area, and piston supposed impervious to heat, contains 100 cubic inches of dry air under a pressure of 14·7 lb. per square inch. What weight must be put on the piston for the air to be reduced to 10 cubic inches in bulk?

6. The heat-proof cylinder and piston of Question 5 contains 100 cubic inches of dry air at 10°C., under a pressure of 76 cm. of mercury. An *additional* pressure equal to 228 cm. of mercury is suddenly applied. What will be the temperature of the compressed air?

ANSWERS.

4. (i.) 5869 foot-pounds; (ii.) 6772·5 foot-pounds.

5. 351·4 pounds.

6. 151·8°.

LESSON XIV.

Heat Engines. Relation of Pressure to Melting Point.

Efficiency. In a steam engine steam is generated under pressure. This steam acts against a piston for a time, and after doing work by pressing on the moveable piston, is condensed in the air or in a cool place. Now from experiment it is found that a pound of carbon when burnt in air will raise 8080 pounds of water one degree centigrade, and this energy, from Joule's experiments, is equal to 8080 × 1390, or 11,231,200 foot-pounds, and it might fairly be supposed that a steam engine if made so that there was no friction would produce this amount of mechanical energy for every pound of coal (carbon) burnt. As a matter of fact, however, only a fraction of the energy of the burning fuel is thus converted. To begin with, heat is lost from the heating of the brickwork of the furnace, the iron of the boiler, and the gases of the chimney, and this ultimately warms the air without producing any useful work; but even assuming that the 8080 pound-units were used entirely in making steam, there will still be loss, as the waste steam is hotter than the condenser into which it is turned, so that heat-energy has to be transferred to the condenser, and is lost as far as the engine is concerned.

"DEFINITION OF EFFICIENCY. *If H is the supply of heat, and W the work done by an engine, both measured in foot-pounds, then the fraction* $\frac{W}{H}$ *is called the efficiency of an engine.*"—Maxwell's *Theory of Heat*, 1877, page 151.

Now it may be shown by what is known as Carnot's principle (see Balfour Stewart's *Heat*, page 340; Clerk Maxwell, *Theory of Heat*, page 138,) that W, the work done by an engine, is dependent only on two things: one, the supply of heat (H), the other, the difference of temperature between the source of heat and the condenser.

With any heat engine we may suppose the work done by the engine to be stored up in the form of potential energy, as, for example, in the form of a raised weight. This weight may be released, and made by means of friction to heat the water or other substance in the boiler. The term "reversible" implies that an engine may be made to store up potential energy, and that this energy may be applied to restore the temperature of the boiler to its original number of degrees above that of the condenser, that is, the engine may be made to work backwards—at one time transferring heat from the boiler to the condenser, and producing mechanical energy, at another time losing mechanical energy, and making the temperature of the source hotter than that of the condenser. A reversible engine which will, if worked backwards, reproduce the original difference of temperature between the boiler (source) and condenser (refrigerator) is termed a "perfect" engine.

The efficiency of a heat engine is the ratio

$$\frac{\text{Heat utilised}}{\text{Heat supplied}}=\frac{W}{H}$$

Thus, if a boiler were supplied with 10,000 pound heat-units, or 10,000 × 1390 foot-pounds, and the mechanical work done by the engine were 139,000 foot-pounds, the efficiency of the engine would be

$$\frac{139{,}000}{10{,}000\times 1390}=\frac{1}{100}, \text{ or 1 per cent.}$$

Now it is shown by Carnot's principle that if the engine be a reversible or perfect one, that

$$\frac{W}{H}=\frac{S-T}{S}$$

where W = mechanical work in foot-pounds, ergs, &c.,
H = heat supplied expressed in foot-pounds, &c.
S = absolute temperature of boiler or source.
T = absolute temperature of condenser or refrigerator.

It may be noticed here that the ratio W/H can only be unity when $T=0$, so that even if a steam engine could be made perfect, the mechanical energy obtained could only equal that of the heat supplied when the temperature of the condenser was at absolute zero. With engines of modern construction the efficiency is not more than 10 per cent. of the energy of the fuel used. A marine engine, whose performance was examined by Professor A. B. W. Kennedy, gave 11·25 per cent. of the heat received by the feed water, and 7·6 per cent. of the heat given out by the fuel used. (See *Engineer*, May 2nd, 1890, page 365.)

EXAMPLE A. A steam cylinder is supplied with steam at a temperature at 110°C., and after passing through the engine is condensed in water at a temperature of 50°C. Assuming the engine to be a perfect one, find its efficiency.

$$\text{Efficiency} = \frac{\text{Heat utilised}}{\text{Heat supplied}} = \frac{W}{H}$$

$$= \frac{S-T}{S} = \frac{(273+110)-(273+50)}{273+110}$$

$$= \frac{60}{383} = 0·157,$$

or efficiency equal 15·7 per cent. In such an engine then 15·7 per cent. of the heat supplied to the boiler is utilised in producing mechanical work, while 84·3 per cent. goes to warm the condenser.

EXAMPLE B. How many pound-units of heat per minute must be supplied to the boiler of the engine in last example for the engine to develop 100 horse-power per minute?

A horse-power is 33,000 foot-pounds of work per minute, and the engine must develop 100 times this; that is, must do 3,300,000 foot-pounds per minute.

Now, from Example A, this 3,300,000 foot-pounds per minute is only 15·7 per cent. of the heat-energy supplied, so the actual heat-energy to be supplied will be

$$3,300,000 \times \frac{100}{15·7}$$

and the equivalent of this in pound-degree heat-units will be

$$\frac{3,300,000 \times 100}{15·7 \times 1390} = 15,122 \text{ nearly.}$$

This shows us, then, that for a 100 horse-power engine 15,122 units of heat per minute must be supplied to an engine if the steam is generated at 110°, and condensed at 50°.

NOTE.—The horse-power of an engine is generally calculated by the formula

$$\text{HP} = \frac{P.\ L.\ A.\ N.}{33,000}$$

where P = mean pressure of steam on piston in pounds per square inch.
L = length of stroke in feet.
A = area of piston in square inches.
N = number of strokes per minute.

The mnemonic word *PLAN* enables one easily to remember this formula. (See *Elementary Treatise on Steam*, Perry. Macmillan & Co., 1874, page 62.)

The *nominal* horse-power is the above formula when P has an assumed value, generally of 7 lb. per square inch. *Indicated* horse-power is the work per minute, as obtained from an indicator diagram.

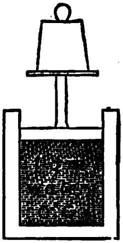

Fig. 52.

Effect of pressure in altering the melting point. Fig. 52 represents a mass of water at 0° enclosed, in an inextensible non-conducting vessel, provided with a tight-fitting weightless piston also of a non-conducting material. On the top of the piston a weight of 1033 gm. is placed, producing a pressure equal to that of the ordinary atmospheric pressure.

If a quantity of heat (*H*) is withdrawn from the water, the water is frozen, and the weight rises. doing an amount of work (*W*) in lifting the weight, and we have by Carnot's principle

$$\frac{W}{H}=\frac{S-T}{S}$$

Let now the engine work backwards, that is, let heat be imparted in order to melt the ice, the weight descends. Now we know the heat (*H*) imparted to melt the ice, we know the work (*W*) done in the falling of the weight; we also know the temperature (*S*) of the ice 0°C., or 273 on the absolute scale. Hence $S-T$, or the difference in temperature between the ice and the water produced, can be calculated. Thus, if the vessel of fig. 52 contains 1 gramme of ice, the heat (*H*) necessary to melt it is 80 calories, or $80 \times 4{\cdot}2 \times 10^7$ Joules. If the piston has an area of one square centimetre, the piston falls 0·087 cm. during the melting of the ice, or the negative work (*W*) done is $1033 \times 981 \times 0{\cdot}087$, or 88,163 ergs; hence

$$\frac{88{,}163}{80 \times 4{\cdot}2 \times 10^7}=\frac{273-T}{273}$$

or $273-T=0{\cdot}00716$.

Hence the temperature of the water produced will be colder than the ice by 0·00716 degree centigrade. The greater the pressure, of course the greater the difference of temperature. If, for example, the pressure were 2066 grammes instead of 1033, then *W* would be twice as great, and instead of 0·00716 we should have double this.

EXAMPLE C. A melted substance has a latent heat of 30 calories per cubic centimetre, and melts under atmospheric pressure at 50°C. The melted substance shrinks 2/10ths of its volume on solidifying. Required to what extent its melting (or solidifying) point will be *raised*, supposing it to solidify under 10^6 dynes additional pressure.

Imagine a cubic centimetre of the substance melted at 50°, and then to solidify, an amount of (negative) work is done equal to that of 10^6 dynes working through 2/10ths of a centimetre, that is, equal to $10^6 \times 2/10$, or 2×10^5 ergs. The amount of heat liberated is 30 calories; hence we have

$$\frac{W}{H}=\frac{2\times 10^5}{30\times 4\cdot 2\times 10^7}, \text{ and } \frac{S-T}{S}=\frac{273+50-T}{273+50}=\frac{323-T}{323}$$

from which
$$323-T=\frac{323\times(2\times 10^5)}{30\times 4\cdot 2\times 10^7}=0\cdot 0511$$

or the melting point will be raised 0°0511C.

EXERCISE XIV.

1. Three steam engines, *A*, *B*, *C*, are in action. In *A* the temperature of the steam is 120°, and that of the condenser 15°. In *B* the temperature of the steam is 102°, and that of the condenser is 15°. In *C* the temperature of the steam is 130°, and of the condenser (–5)°. Assuming the engines reversible, as by Carnot's principle, find the efficiency of each, per cent., to one place of decimals.

2. From the following table calculate the horse-power to the nearest whole number of each of the engines *A*, *B*, *C*.

	Mean Pressure on Piston in lb. per square inch.	Length of Stroke in feet.	Area of Piston in square inches.	No. Strokes per minute.
Engine *A*	20	6	400	60
„ *B*	8	11	3600	40
„ *C*	25	3	100	90

3. From the efficiencies found for the engines *A*, *B*, *C*, in Example 1, and the horse-power as found from Example 2, determine how many pound-units of heat per minute must be supplied to the boiler of each engine.

4. An imaginary solid has at its melting point (200°C.) a volume of 100 cubic centimetres, but when melted a volume of 110 cubic centimetres. Will its melting point be raised or lowered when submitted to an increased pressure of 10^6 dynes per sq. cm.? Taking the latent heat of the liquid as 100 calories per cubic centimetre of the solid, find to what extent the melting point will be affected when it is submitted to a pressure of 10^6 dynes per sq. cm.

ANSWERS.

1. *A*, 26·7; *B*, 23·2; *C*, 33·5.
2. *A*, 87; *B*, 384; *C*, 20.
3. *A*, 7736; *B*, 39,500; *C*, 1417 pound-units.
4. 0·01126°C. Raised.

LESSON XV.

The Kinetic Theory of Gases.

Introduction. To make intelligible the fact that all the permanent gases behave in the same way when subjected to variation of pressure and temperature, Avogadro propounded the hypothesis that all gases contain the same number of molecules. Thus one may on this hypothesis understand why all gases behave alike, because the change of pressure, or variation of temperature, simply affects the intermolecular space, which is the same for all gases. Just as two series of balls of different materials connected together with springs of the same material will be elongated or compressed by equal variations of pressure, so a space containing molecules of hydrogen, oxygen, hydrochloric acid gas, &c., will all behave alike if that which separates the unlike molecules is the same. But this static idea of the molecular constitution of a body is not sufficient to explain other phenomena than that of equal expansion. Gases diffuse rapidly into each other, obeying Graham's law,* and a conception of equal repulsion of different molecules will not satisfactorily account for this. But a more serious difficulty is the experiment of Joule, referred to on page 103, by which it is shown that the mere separation of molecules does not require that work, positive or negative, shall be done.

The kinetic theory of gases, when once the conception is granted, explains however in an extremely elegant manner the whole of the phenomena referred to. What, then, is this conception? All gases are regarded as made up of exceedingly minute particles identically equal in the same gas, though different in different gases, called molecules, and these molecules are in rapid motion. The velocity with which the molecules move is constant so long as the temperature is constant, but immediately the temperature is raised the velocity quickens, or if lowered the velocity slackens. This in broad outline is the conception.

The primary difficulty of the student is to imagine such a state of things as the conception requires. Suppose a cubic glass box, which measures 1 millimetre along one inside edge, to contain a gas, hydrogen, for example, under a pressure of 76 cm. of mercury, and at a temperature of 0°C., and let this box be put under a microscope of such power that the molecules of hydrogen and the molecules of glass could easily be seen. To enable the observer to follow the motion of the molecules, which would of course be invisible except when moving directly towards or from the observer, a device similar to Savart's rotating wheel may be interposed. What, then, according to the theory, would be seen?

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* See *Arithmetical Chemistry, Part II.*, page 63.

We should see millions of molecules of hydrogen moving about in all directions, "like bees in a box," frequently striking each other, but rebounding much as billiard balls would do, now bumping against the molecules of glass which confine them, but receiving from the glass bumps in return. Some molecules would be moving slowly, some rapidly, some with medium velocity, but it would be observed that the average velocity of all the molecules would be the same if followed for a short interval of time.

While the box of molecules still retains the same temperature, let its volume be reduced to one-half. Then looking through our microscope the only difference we shall observe is that the same number of molecules are moving about with the same speed, but in a box of half the volume. The number of blows of the molecules on the sides of the box will be just doubled, in consequence of this double crowding of the molecules.

Now restore the box to its former size and condition, that is, have a cubic millimetre box of molecules at 0°C. and 76 cm. pressure, and, while looking through the microscope, let the box and its contents be warmed. We should now note that while of necessity the number of molecules in the box remains the same, the motion of each molecule is quickened, and in consequence of this quickening the sides of the box are struck more frequently and more forcibly, and as a result there is an increased pressure on the sides of the box.

Had our cubic millimetre box contained oxygen molecules instead of hydrogen molecules, the same general phenomena would have been observed, the only difference consisting in a slower motion of the oxygen molecules.

The student may at first form an objection to this view, inasmuch as he would expect the molecules to come to rest; but on reflection he will see from the doctrine of energy that they cannot alter their motion as long as the temperature remains the same.

Billiard balls come to rest after striking the cushion and each other several times, but the temperature of the billiard table is raised directly there is any fall of motion of the balls. If the cushion were perfectly elastic, the balls also, the table smooth, and the air absent, then once started the balls would never come to rest, though they would produce pressure on the cushions.

Let us now put the above ideas in quantitative form.

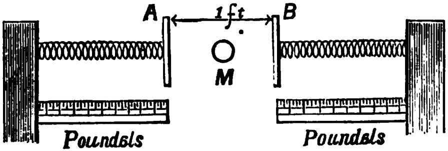

Fig. 53.

Pressure of a Gas. Consider in the first place what pressure will be produced by a ball which moves about between two elastic walls. In fig. 53 is represented an imaginary experiment. The ball *M* has a mass of 1 lb., and oscillates between two moveable walls, *A* and *B*. These walls form the pans of two spring balances, as represented in the figure. Graduations on rulers below the pans express the pressure in poundals, that is, in this part of the world. the ordinary pressure multiplied by about 32. The distance between the pans when at rest is 1 foot. Further. the experiment is supposed to be made in a vacuum, where $g=0$, so that the ball has no tendency to fall, and the ball, as well as the walls, are perfectly elastic.

Let *M* have a velocity of 1 foot per second, and strike the pan *A*, the spring of *A* is compressed, the ball gradually comes to rest as compression takes place; the spring now extends, and finally shoots the ball against the pan *B*, which returns it, and so the ball oscillates from one pan to the other, producing pressure on each pan, which can be registered by the index. We want now to ascertain, with these conditions, what will be the mean pressure per second that the springs will register.

Let us assume that each spring is tempered so delicately that it takes just one second to stop the ball, and, of course, another second to give it full impetus in the opposite direction. Then the ball is in contact with *A* for 2 seconds; on first touching *A* the pressure is 0, but the pressure gradually increases as the motion of the ball is stopped, until, finally, at the end of 1 second, the index records a pressure of 2 poundals. Hence, while the ball is in contact with *A* (1 second), and moving from right to left, there is a mean pressure of $\frac{1}{2}(0+2)$, that is, 1 poundal.

Similarly, as the spring recovers itself, it starts with a pressure of 2 poundals, and at the moment the ball leaves *A* the pressure is 0; so again the mean pressure during this second second is $\frac{1}{2}(2+0)$, or 1 poundal.

From the above course of reasoning we learn that with a ball of unit mass, and having unit velocity, that is, having unit momentum, we obtain a mean pressure of 1 unit lasting for 2 seconds, which, as we now proceed to show, is the same thing as a mean pressure of 2 units lasting for 1 second; that is, unit momentum produces 2 units of pressure.

If the spring connected with *A* is now tempered so as to occupy 1 second instead of 2 in receiving and returning the ball, the only difference will be that the pressure recorded will be twice as great as before; or the scale will register 4 poundals instead of 2, and the average pressure will be $\frac{1}{2}(0+4)$ lasting half a second, and $\frac{1}{2}(4+0)$ lasting another half-second; or, together, a mean pressure of 2 poundals, lasting for 1 second. And it is pretty clear that if we go on increasing the temper of the springs, the effect is to increase the final pressure recorded, but at the same time to diminish the time during which it lasts; but, whatever may be the rigidity of the springs, the mean pressure will be always equal to 2 poundals lasting for 1 second.

Let, then, the springs be made so rigid that the time occupied by the spring *A* in receiving and returning the ball is so small that it may be disregarded, the pressure is still 2 poundals per second produced by the blow of the ball *M*. Of course the same holds in reference to the pan *B*. As the ball is oscillating from one pan to the other, and the distance between the pans is 1 foot, the pan *P* will be struck at every second second, so that the pressure communicated to the pan *A* is 2 poundals every 2 seconds, or 1 poundal per second. Similarly the pressure communicated to *B* is

1 poundal per second. If the ball had a mass of 2 pounds instead of 1 pound, then the pressure would be doubled, or generally the pressure varies as M varies.

Let us see now the effect of increasing the speed of the ball. Suppose it is two feet per second, the momentum of each blow is twice as great, and the number of blows twice as many; hence the mean pressure produced per second is now four times as great, or the mean pressure is 4 poundals lasting for each second of time.

Put now the result in a general form. If M is the mass of the ball, and V its velocity in feet per second, then its pressure per second on the pan A is, from the above reasoning, MV^2 poundals.

If we have a cubic foot of any gas under a pressure P poundals per square foot, then we may imagine the molecules of the gas divided into three equal portions: one portion moving right and left, one portion moving to and fro, and one portion moving up and down. Each of these portions produces the pressure P. Now if M is the mass of a single molecule of the gas, and N is the number of molecules in one cubic foot, then NM is the mass of a cubic foot of the gas, that is, its density d. Hence, from what has been said, we have

$$\tfrac{1}{3}NMV^2 = \tfrac{1}{3}dV^2 = P \text{ (i.)}, \quad \text{or } V = \sqrt{\frac{3P}{NM}} = \sqrt{\frac{3P}{d}} \text{ (ii.)}$$

Thus if we are dealing with hydrogen at 0°C. and 2116·8 lb. pressure per square foot, at a place where $g = 32{\cdot}2$, then since NM, or the mass of a cubic foot, $= 0{\cdot}0056$ lb., we have

$$\tfrac{1}{3} \times 0{\cdot}0056 \times v^2 = 2116{\cdot}8 \times 32{\cdot}2$$

$$\text{or } V = \sqrt{\frac{3 \times 2116{\cdot}8 \times 32{\cdot}2}{0{\cdot}0056}} = 6043 \text{ ft.}$$

Hence we learn, as first shown by Joule, that the velocity of the molecules of hydrogen, according to the molecular theory, must be 6043 ft. per second, if the gas is at 0°C., and $2116{\cdot}8 \times 32{\cdot}2$ poundals pressure.

If we use the C.G.S. units, then we must express density as the mass of a cubic centimetre of the gas under the pressure P in dynes.

From equation (ii.) we can see, since NM or the density d of a gas varies directly as P the pressure upon it, that V will remain the same for the same gas at the same temperature whatever the pressure, the only effect of pressure being to crowd more molecules into the same space.

Again, if we have two gases which have densities $d_1 d_2$, at the same temperature and the same pressure P, then if V_1 is the velocity of the molecules of the first gas, and V_2 the velocity of the molecules of the second gas, then

$$V_1 = \sqrt{\frac{3P}{d_1}}$$

$$V_2 = \sqrt{\frac{3P}{d^2}}$$

hence, dividing, we have

$$\frac{V_1}{V_2} = \frac{\sqrt{d_2}}{\sqrt{d_1}} \qquad \text{(iii.)}$$

or the velocities of the molecules of two gases at the same temperature and pressure are inversely as the square roots of their densities. Thus if V_1 refer to hydrogen $d_1 = 1$, and V_2 to oxygen $d_2 = 16$, then $V_1 : V_2 :: 4 : 1$, which accounts for Graham's *Law of Diffusion.*

Let us now consider the case of the same gas at different temperatures, and find effect of temperature in altering the velocity of molecules.

The student is well aware that if d_1 is the density of a gas at a temperature t_1, then its density d_2 at a temperature t_2 is obtained by multiplying d_1 into $273 + t_1/(273 + t_2)$. Thus the density of hydrogen at 0°C. being 0·0056 lb. per cubic foot, its density at 100°C. will be $0{\cdot}0056 \times (273 + 0)/(273 + 100)$, that is, 0·0041 lb. per cubic foot. If we express the above in formula (iii.), we have

$$\frac{V_1}{V_2} = \frac{\sqrt{d_2}}{\sqrt{d_1}} = \frac{\sqrt{273 + t_1}}{\sqrt{273 + t_2}}$$

or the velocity of the molecules of a gas increases generally as the temperature of the gas rises, ***being directly proportional to the square root of the absolute temperature.***

Kinetic energy of a gas. According to the theory we are now developing, pressure is produced by the motion of a number of elastic molecules. To put these molecules in motion work has to be done. The question is how much work must be done in quickening the motion of the molecule when the gas rises through 1°C.

If a mass m has a motion V its kinetic energy (as explained in Chapter I. of *Arithmetical Physics, II*B) is $\frac{1}{2}mV^2$; hence, as NM is the mass of a cubic foot of any gas, the kinetic energy of a cubic foot of *any* gas is

$$\tfrac{1}{2}NMV^2$$

but as $NMV^2 = 3P$ (i. page 126), we have $\frac{1}{2}NMV^2 = {}^3/_2P$, or

$$\left.\begin{array}{l}\text{Kinetic energy in foot-poundals of a cubic foot of any}\\ \quad\text{gas where } P \text{ is pressure in poundals per square foot}\end{array}\right\} = \tfrac{3}{2}P$$

Hence, to find the work that has been done in setting the molecules of a cubic foot of any gas in motion, we find what pressure in poundals per square foot the gas is supporting, and one and a half times this is the required work.

We find, then, that 3175·2* foot-pounds of work must be done to put the molecules in the agitation they have at 0°C., or 273° on the absolute scale; for one degree the work will be 1/273rd of this, or 11·6 foot-pounds approximately.

Since the kinetic energy of a cubic foot of *any* gas is the same, it follows, from the above reasoning, that the work required to raise a cubic foot of any gas through 1°C. is 11·6 foot-pounds, so far as the mere agitation of the molecules is concerned in producing pressure.

If the molecules of gases were single points, and so incapable of internal motion, doubtless this 11·6 foot-pounds would be the work required to raise all gases 1°C.; but chemical investigations show that this is not the case, thus hydrochloric acid molecules must contain at least two parts (hydrogen and chlorine), nitric acid five (nitrogen, hydrogen, and three atoms of oxygen), and so on; and it seems reasonable to conclude that, since the molecules are made up of parts, these parts are moving about each other, rotating, vibrating, or exhibiting some kind of *internal energy.*

---

* **That is 1½ times the ordinary pressure of 2116·8 lb. per square foot.**

Now, knowing the specific heat at constant volume, that is, the heat-energy requisite to raise a unit mass of gas 1°C., we can easily find the total energy in foot-pounds requisite to raise a unit volume of gas 1°C., and subtracting 11·6 from this, we have the interior work which has been done upon the molecules.

Thus the specific heat of hydrogen at constant volume is 2·4, and the heat required to raise a cubic foot of hydrogen weighing 0·0056 lb. 1°C., will be 0·0056 × 2·4, or 0·01344 heat-units, and since one heat-unit = 1390 foot-pounds, we have the energy in foot-pounds to raise one cubic foot of hydrogen 1°C. = 0·01344 × 1390, or 18·7 foot-pounds.

We see then

Total work done in raising a cubic foot of hydrogen 1°C., at constant volume ... ... } = 18·7 foot-pounds.

Work required to agitate the molecules, on supposition that no interior work is performed ... } = 11·6 foot-pounds.

Therefore, interior work done in raising the cubic foot of hydrogen 1°C. ... ... } = 7·1 foot-pounds.

Of the whole work done, then, in warming the gas 1°C., while 11·6 parts are devoted to producing pressure, 7·1 parts are devoted to interior work, or if one part is devoted to pressure, 7·1/11·6, or 0·6 about, is devoted to interior work.

Now, with all the permanent gases, the specific heat at constant volume of a unit volume of gas is the same; hence we may say

Kinetic energy in unit volume of any permanent gas $=\frac{1}{2}\beta NMV^2$ where $\beta = 1{\cdot}6$ about, NM = mass of unit volume of the gas, and V the average velocity of its molecules; or we may say (since $NMV^2 = 3P$) that the kinetic energy of a unit volume of a permanent gas is $\frac{3}{2}\beta P$. But as the kinetic energy of unit volume of a gas is, from chemical considerations, the kinetic energy of a molecular volume of the gas, we have the law of Dulong and Petit, as applied to gases, which states *that the molecular weight multiplied into the specific heat is a constant quantity.*

Although β is about 1·6 with the permanent gases, with others it is greater. It is shown on page 319 of Clerk Maxwell's *Theory of Heat* that

$$\beta = \tfrac{2}{3}\,\frac{1}{\gamma - 1} \qquad \text{(iv.)}$$

where γ = the ratio of specific heat at constant pressure to specific heat at constant volume. Thus with ether $\gamma = 1·059$; hence $\beta = 2/3 \times 1/0·059 = 11·3$, so that with ether, when its temperature is raised, one part of the energy is consumed in increasing the pressure of the ether molecules against the sides of the vessel containing them, while 10·3 parts are consumed in the interior work, such as rotating the molecules. From the table among the constants, at end of book, it will be noticed that generally the more complex the molecule the lower is the ratio γ, and hence the higher will be the value of β; that is, the more complex the molecule the greater the amount of energy absorbed in interior work.

If in equation iv. γ be unity, that is, if the specific heat at constant volume equal specific heat at constant pressure, then β would be infinity, and we should have the case of an infinitely complex molecule. On the other hand, if no interior work is done in warming a gas, β must = 1, so that $\frac{1}{2}\beta NMV^2$ shall $= \frac{1}{2}NMV^2$; but for β to be 1 in equation iv., γ must be 5/3, or 1·66, and this is the actual value Kundt found for mercury vapour—which of all vapours seems, from chemical considerations, to be the simplest. (See page 109.)

Suppose we have a cubic foot of a gas *A* containing N_1 molecules, each of mass M_1, and having an average velocity V_1, and by the side of this let there be a cubic foot of another gas *B*, having N_2 molecules each of mass M_2 and velocity V_2; then if both gases are at the same temperature and pressure, we have from page 128

$$\tfrac{1}{2}N_1M_1V_1^2 = \tfrac{1}{2}N_2M_2V_2^2 \qquad (a)$$

Now let the two gases be mixed so that the molecules of *A* bump against the molecules of *B*. Then if a single molecule of the one gas has more kinetic energy than a single molecule of the other gas, exchange takes place, and, finally, the kinetic energy of each single molecule of each gas is the same, that is,

$$\tfrac{1}{2}M_2 V^2 = \tfrac{1}{2}M_2 V_2{}^2 \qquad (b)$$

Divide equation *a* by equation *b*, and we have

$$N_1 = N_2$$

that is, the number of molecules in unit volume is the same for the gas A as for the gas B, and as the symbols used are perfectly general, the above reasoning shows *that all gases contain the same number of molecules in the same volume.* (Avogadro's law.)

Again, since the number of molecules in unit volume is the same for all gases, it follows *that the molecular weights of gases are proportional to the densities* of the gases. (*Gay-Lussac's Law.* Maxwell.)

General remarks. *Absolute zero* will now be understood as a temperature at which the molecules of a gas would have no motion.

It would seem that if the atmosphere consists of a multitude of solid molecules, it should obstruct light much more than it does. Even mercury vapour is transparent. According to Maxwell (*Journal Chemical Society*, vol. XIII., page 506), certain theoretical investigations of Lord Rayliegh may explain this difficulty.

Unless a gas is highly rarefied, its molecules knock against each other frequently, and there is scarcely any free path. As the gas is more and more rarefied, the free path of a molecule becomes longer, and "during its free path it vibrates according to its own laws, the amplitudes of the different simple vibrations being determined by the nature of the collision, but their periods depending only on the constitution of the molecule itself. If the molecule is capable of communicating these vibrations to the medium in which radiations are propagated, it will send forth radiations of certain definite kinds, and if these belong to the luminous part of the spectrum, they will be visible as light of definite refrangibility. This, then, is the explanation, on the molecular theory, of the bright lines observed in the spectra of incandescent gases."—Maxwell, *Theory of Heat*, 1877, page 326.

I don't know whether the following analogy will be considered too absurd. A variety of the pigeon tribe called "tumblers" have the peculiarity of flying forward, and every now and then turning over and over for a short time, and then subsequently continuing their flight. Let a multitude of such pigeons be confined within a small enclosure. The birds will fly about, bumping against each other; occasionally a bird may have the chance of tumbling, but from the crowding together there will be no regularity in the tumbling, each bird's rhythmical motion being interfered with by bumps from its neighbour. If the tumbling motions could be sufficiently rapid, the air would be set in vibration, and tones produced. While the birds are crowded together the tones will be of every possible pitch, but if the space within which the birds are confined is now enlarged, a few of the birds will have a free path, and the particular note or notes, due to the natural rhythm of the bird's tumble, will begin to be heard. If the space be still further enlarged, more of the birds will have a chance of tumbling without interference from their neighbours; and, ultimately, we may arrive at a condition in which all is regularity, and nothing but the note or notes due to the rhythm of the tumbles will be heard.

Transfer this idea to molecules. The forward or direct path of the molecule corresponds to the forward flight of the pigeon, while the tumbling will correspond to the vibrating period of the molecule. When the molecules have a free space, as in a Crooke's vacuum, the periods assert themselves, the ether is set in motion, and particular wave lengths of light are emitted, giving the lines of the spectrum. When crowded together, as the molecules are under pressure, the encounters destroy the rhythmic motion, and all wave lengths are produced, giving, according to the pressure, a banded or a continuous spectrum.

EXAMPLE A. Find the kinetic energy of a cannon ball of 10 kilogrammes mass, and moving with a velocity of 500 metres per second; that is, find the work that must have been done by the gunpowder in imparting motion to the ball.

The kinetic energy is $\frac{1}{2}mv^2$; that is, using *C.G.S.* units

$$\frac{1}{2}\times 10{,}000\times(50{,}000)^2=\frac{25}{2}\times 10^{12}=1{\cdot}25\times 10^{13}\text{ ergs.}$$

EXAMPLE B. A gas *A* has a density of 0·08 lb. per cubic foot at 10°C. and 30 in. mercurial pressure when $g=32$ poundals. Find velocity per second of the molecules of this gas. (1 c. in. mercury$=$0·491 lb.)

Note that pressure on each square inch $=0{\cdot}491\times 30$, or $14{\cdot}73$ lb.; or on each square foot $=14{\cdot}73\times 144=2121{\cdot}12$ lb. per square foot, that is, $2121{\cdot}12\times 32$, or $67{,}875{\cdot}84$ poundals per square foot.

Now from ii., page 126, $V=\sqrt{\frac{3P}{d}}$

Hence $V=\sqrt{\left(\frac{3\times 67{,}875{\cdot}84}{0{\cdot}08}\right)}=1595$ ft. per second approximately.

EXAMPLE C. A gas B has a density of 0·00143 gm. per cubic centimetre under a pressure of 1,000,000 dynes (a little over a kilogramme per sq. cm.) and a temperature of 15°C. Find the velocity of the molecules of the gas B at 15°C.

$$V=\sqrt{\left(\frac{3\times 1{,}000{,}000}{0{\cdot}00143}\right)}=45{,}802 \text{ cm.}$$

EXAMPLE D. Supposing it possible for the kinetic energy (exterior work only) of a cubic foot of a gas at 20°C. and 2116·8 pounds pressure per square foot, at a place where $g=32{\cdot}2$, to be transferred to a leaden bullet weighing 0·25 lb., with what velocity would the bullet have to move?

From page 128 the kinetic energy is one and a half times the pressure on the gas.

Kinetic energy $=2116{\cdot}8\times 32{\cdot}2\times(3/2)$. Hence if $v=$ velocity of the bullet $\frac{0{\cdot}25}{2}v^2=\frac{2116{\cdot}8\times 32{\cdot}2\times 3}{2}$

From which $v=904{\cdot}4$ ft. per second.

EXERCISE XV.

1. (*a*) If a mass of 10 lb. fall from a height of 1000 feet at a place where $g=32$, what is its kinetic energy in foot-poundals? What in foot-pounds? (*b*) If a mass of 10 grammes fall from a height of 1000 centimetres at a place where $g=1000$, what is its kinetic energy in ergs?

NOTE.—$v^2=2gs$. Kinetic energy $=\frac{1}{2}mv^2=\frac{1}{2}m\times 2gs=mgs$. (See *Arithmetical Physics*, *II*B, Chapter I.) Hence kinetic energy is found by multiplying the mass into the value of gravity into the vertical height fallen.

2. The mass of a cubic foot of hydrogen at 0°C. and 30 in. mercurial pressure is 0·0056 lb. What is the mass of a cubic foot of the gas at the same pressure, but at temperatures (*a*) of 273°C., (*b*) of 819°C., (*c*) of −136·5°C.?

3. Having found by the previous exercise the mass of a cubic foot of hydrogen at each of the temperatures named, find now the velocity in feet per second of the hydrogen molecules at each temperature, taking the velocity at 0° as 6043 ft., as found on page 126.

4. An imaginary gas has a density of 1 gm. per cubic centimetre, at 0°C., and under a pressure of 1000 cm. of mercury, at a place where g is 1000 dynes. Find the velocity of the molecules of this imaginary gas. (Mass of 1 c.cm. of mercury $=13{\cdot}6$ gm.)

5. Of 3 balls, *A*, *B*, and *C*, *A* has a mass of 1 gramme and a velocity of 20,000 centimetres per second, *B* has a mass of 16 grammes, and *C* has a mass of 36 grammes. With what velocity must *B* and *C* move so that they may each have the same kinetic energy as *A*?

6. If it were possible to transfer all the energy of exterior work contained in a cubic foot of a gas (hydrogen, for example) at 0°C., so as to impart velocity to a mass of 1 lb. of lead, at what rate per second would the mass of lead move?

7. If all the kinetic energy (exterior work only) of a cubic foot of hydrogen at 0°C. could be used to warm water, through what temperature centigrade would a cubic foot of water, that is, 62·4 lb. of water. be raised by this energy? Would it make any difference if oxygen had been mentioned instead of hydrogen?

8. A mass of lead of one pound is set spinning, and the kinetic energy of the spinning mass is 320 foot-poundals. The spinning mass is now shot forward with a velocity of 100 feet per second. What is the total kinetic energy of the moving mass of lead?

9. Find the value of β for oxygen, water vapour, and olefiant gas, taking the value of γ for each gas from the table of constants. From your values say what fraction of the kinetic energy of the gas represents interior and what exterior work.

ANSWERS.

1. (*a*) 320,000 foot-poundals, 10,000 foot-pounds; (*b*) 10^7 ergs, or 10^4 gramme-centimetres.
2. (*a*) 0·0028 lb.; (*b*) 0·0014 lb.; (*c*) 0·0112 lb.
3. (*a*) 8545; (*b*) 12,086; (*c*) 4273 feet per sec.
4. 6387 cm. per sec.
5. 5000, 3333·3 cm. per sec.
6. 452·2 ft. per sec.
7. 0·037°C. nearly. No difference.
8. 5320 foot-poundals.
9. Oxygen 1·654, water vapour 2·207, olefiant gas 4·629; 1 : 0·654, 1 : 1·207, 1 : 3·629.

REFERENCE CHAPTER A.

Definitions and Propositions to which reference is made in the body of the work.

Simple geometrical propositions. *Parallel lines.* If *ab*, *cd*, fig. 54, are parallel lines, and *ef* fall upon them, the angles A_1, A_2, A_3, A_4, are equal to each other, and also the angles B_1, B_2, B_3, B_4, are equal to each other. A_1, B_1, A_4, B_4, are called exterior angles; A_2, A_3, are called alternate angles, as are also B_2, B_3. Any line parallel to *ab* is parallel to *cd*, and similarly any line parallel to *cd* is parallel to *ab*. (See Euclid I., 29, &c.)

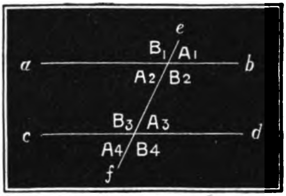

Fig. 54.

Triangles. When in two triangles the sides *bc* and *b′c′*, fig. 55, are equal each to each, and the included angles *A*, *A′*, are also equal, then the triangles are equal in every respect. Also if the three sides are respectively equal, that is, if $a=a'$, $b=b'$, and $c=c'$, then the triangles are equal in every respect.

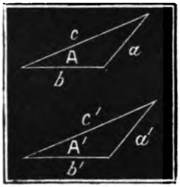

Fig. 55.

The three angles of a triangle, usually denoted *A*, *B*, *C*, as in fig. 56, but left out in fig. 55 for want of space, are together equal to two right angles, that is, $A+B+C=180°$.

In a *right-angled* triangle with *C*, fig. 56, as a right angle, *c* is the hypothenuse, *a* the perpendicular, and *b* the base. The angle $A=(90°-B)$, and the angle $B=(90°-A)$, also $c^2=a^2+b^2$. (See Euclid I., 33 and 47.)

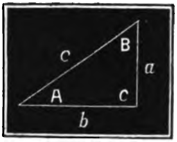

Fig. 56.

Two triangles which are equi-angular have their sides proportionals. Thus, if the triangles of fig. 55 are no longer equal as regards sides, but have two of their angles equal each to each, $A=A'$, and $B=B'$, and (of necessity) $C=C'$. then no matter what may be lengths of the sides a, b, c, we shall have

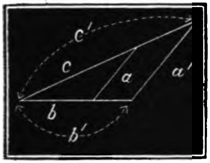

Fig. 57.

$$\frac{a}{b}=\frac{a'}{b'}, \quad \frac{b}{c}=\frac{b'}{c'}, \quad \frac{c}{a}=\frac{c'}{a'} \qquad (A)$$

also if within a triangle a line is drawn parallel to one of the sides, we have proportional lines. Thus, in fig. 57, if a and a' are parallel, the three equations (A) still hold. (Euclid VI., 4.)

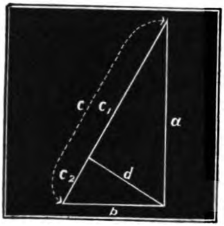

Fig. 58.

If a right-angled triangle abc, fig. 58, be divided into two right-angled triangles by the line d (which, of course, is perpendicular to c), then the three triangles abc, adc_1, and bdc_2, are similar, and their sides proportionals, that is,

$$a:b:c=c_1:d:a=d:c_2:b$$

(Euclid VI., 8.)

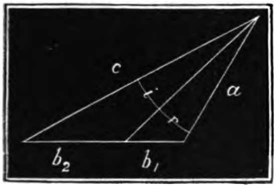

Fig. 59.

If in a triangle a line be drawn bisecting one of the angles, the segments of the base are proportional to the sides. Thus, if the angles i and r, fig. 59, are equal, then

$$\frac{a}{c}=\frac{b_1}{b_2} \quad \text{(Euclid VI., 3.)}$$

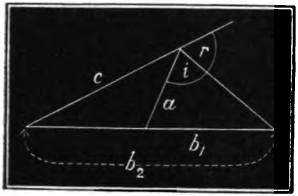

Fig. 60.

If the outward angle made by producing one of the sides of a triangle be divided into two equal angles by a line which meets another side produced, we have proportionals. Thus, if angles i and r, fig. 60, are equal, then

$$\frac{a}{c}=\frac{b_1}{b_2} \qquad \text{(Euclid VI., 3A.)}$$

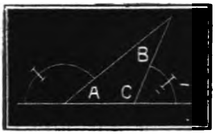

Fig. 61.

The exterior angle I, fig. 61, of a triangle is equal to the angles B and C together, and the exterior angle I' is equal to the angles A and B together. (Euclid I., 32.)

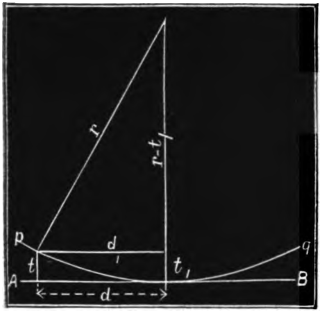

Fig. 62.

If a circle pq touches a line AB in the manner indicated in fig. 62, then

$$t=\frac{d^2}{2r}$$

where r is radius of the circle, t the vertical distance between AB and the extremity of the radius r, and d is the horizontal distance between the line t and the point where the circle touches the line.

For $$(r-t_1)^2=r^2-d_1{}^2$$
or $r^2-2rt_1+t_1{}^2=r^2-d_1{}^2$. Now $t_1{}^2$ is so small it may be struck out.

Hence $d_1{}^2=2rt_1$, or since $d=d_1$ and $t=t_1$

$$d^2=2rt, \text{ and}$$

$$t=\frac{d^2}{2r}$$

The figure may be regarded as the section of a sphere resting on a plane, as in Newton's rings experiment. Some problems on this formula are given in Exercise VIII., page 82.

Trigonometrical Ratios. An angle is the opening between two straight lines which meet together. If, from any point in one of the lines, a perpendicular fall upon the other line, we have the right-angled triangle, as in fig. 63. Then, if A is the angle enclosed by the straight lines expressed in degrees,

$$\frac{a}{c}=\sin A, \quad \frac{b}{c}=\cos A, \quad \frac{a}{b}=\tan A$$

Since a/c divided by $b/c = a/b$, we have

$$\tan A = \frac{\sin A}{\cos A}$$

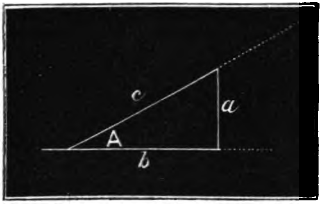

Fig. 63.

It is immaterial at what point the perpendicular is allowed to fall. If, however, we take c as equal to 1 inch, 1 centimetre, or any unit of length, then the various ratios may be interpreted as lengths. Thus, if A is 55°, $\sin A = 0{\cdot}819$, $\cos A = 0{\cdot}574$, $\tan A = 1{\cdot}428$, which may be interpreted as meaning that with a triangle, such as fig. 63, that if c is one inch long, and A is 55°, then a is 0·819 inch, b is 0·574 inch; and a is 1·428 times as long as b, that is, it is $1{\cdot}428 \times 0{\cdot}574$, or 0·820 inch nearly.

EXAMPLE A. An observer is 100 feet from a spire, and observes by means of a theodolite that the summit of the spire is 35° above the horizontal line. Required the height of the spire. The perpendicular a, fig. 63, may represent the spire, then we have from the problem that b is 100 feet, and A is 35°. Looking to the table of trigonometrical functions on cover, we see that $\tan 35° = 0{\cdot}700$; so that a is 7/10ths of b, and as b is 100 ft, a is $100 \times 7/10$, or 70 feet. The spire, then, is 70 feet above the eye of the observer.

The sides of triangles are proportional to the sines of opposite angles. Thus in any triangle

$$\frac{\sin A}{\sin B} = \frac{a}{b}, \quad \frac{\sin B}{\sin C} = \frac{b}{c}, \quad \frac{\sin C}{\sin A} = \frac{c}{a}$$

If the triangle, as in figure 56, is right-angled, then, as $\sin C = \sin 90° = 1$, we have

$$\sin A = \frac{a}{c}, \quad \sin B = \frac{b}{c}, \quad \sin C = 1$$

Circular Measure. To the non-mathematical student, angles in degrees are generally better understood than angles expressed in circular measure.

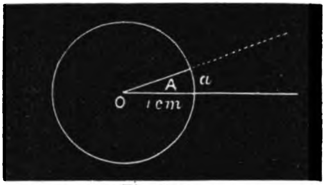

Fig. 64.

Let it be required to express the angle A, fig. 64, in circular measure. About the point O describe a circle having a radius of 1 cm.

Measure the length of the arc *a* in centimetres, then

$$\text{circular measure of } A = \frac{a}{1}$$

If the arc *a* is just 1 cm. long, the measure of the angle *A* is 1/1, or 1; if the arc is 2 cm., the measure is 2; if it is 3·1416 ..., or π cm., that is, half round the circle, the measure is π, or 180°; if the arc *a* is taken as all round the circle, or 2π cm., the angle is 360°.

From this explanation the student can easily convert one kind of measure into the other, for if x is circular measure of an angle, and A° represents any angle in angular measure,

$$360^\circ = 2\pi \text{ in circular measure.}$$

$$A^\circ = x \quad ,, \quad ,, \quad ,,$$

or

$$x = \frac{2\pi A^\circ}{360^\circ}$$

Thus the circular measure of $36^\circ = \frac{2\pi \times 36}{360}$, or $\frac{\pi}{5}$

The term *radian* has been suggested to denote an angle equal in circular measure to unity. Thus, the above angle would be expressed as $\pi/5$ radians. A radian is 57·29578 degrees. Hence circular measure is converted into degrees by multiplying the circular measure by 57·29578.

The student should familiarize himself with the mode of finding the values of trigonometrical functions, and conversely the method of finding the angle from a given function. The directions with the table on cover are sufficient for the purpose.

Positive and Negative Direction of Rotation. If in fig. 64 the line marked 1 cm. remain fixed, whilst the other line turns about the centre *O*, so as to increase the angle *A*, then the radius turns in the *positive* direction. If, on the other hand, the radius turns, making the angle *A* smaller, the radius turns in the *negative* direction. When the radius coincides with the horizontal line, the angle is 0°; if it continue turning, we have an angle below the horizontal line—this is a negative angle. The hands of a watch turn in the *negative* direction. The ordinary corkscrew, when entering a cork, turns in the negative direction, but on unscrewing it turns in the positive direction.

If we assume that at a quarter past the hour the hour hand is on the figure corresponding to the hour, then the angle between the minute hand and hour hand, *measured from the minute hand*, will be

Time.	Angle.	Time.	Angle.
3h. 15m.	0	8h. 15m.	+210°, or −150°
2h. 15m.	+ 30°	6h. 15m.	+270°, or − 90°
1h. 15m.	+ 60°	3h. 15m.	+360°, or −360°, or 0
10h. 15m.	+150°		
9h. 15m.	±180°		

Representation of Points and Lines in one Plane by reference to Axes. Rule on a drawing board two lines, XX', YY', at right angles to each other, fig. 65, and crossing at O. Then any point, such as either P_1, P_2, P_3, P_4, may be defined by stating that it lies a certain distance to the right or left of YY', and a certain distance above or below XX'. In defining the position of a point, then, we want two numbers, expressing the lengths of two lines, and also some conventional expressions to indicate the directions in which these lines are to be drawn. The length of line having reference to distance from YY' is named first, and + (expressed or understood) means that the line is drawn to the right of YY', while the sign − indicates that the line is drawn to the left of YY'. The same convention is used in reference to the second number, but the + sign in this case signifies above XX', while the minus sign signifies below XX'.

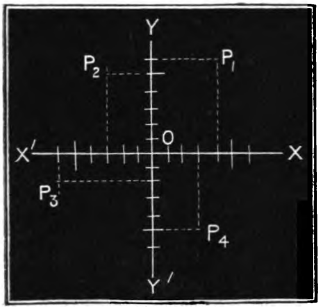

Fig. 65.

Thus the point P_1, of fig. 65, is fully expressed by calling it the point 4 6, for this means that the point is found by drawing a line along OX to a distance 4, and then a second line to a distance 6 above OX. The point P_2 is expressed as $\bar{3}$ 5, the distance along OX being to the left, or minus, while the distance from OX is still above, or positive. Similarly P_3 is $\bar{6}$ $\bar{2}$, and P_4 is 3 $\bar{5}$.*

---

* Other ways of expressing the points are (4, 6), (−3, 5), (−6, −2), (3, −5).

The Indicator Diagram. As explained in the next chapter, work is measured by the product of the pressure applied into the distance through which the pressure acts; and this work is very conveniently given in the form of a diagram termed the indicator diagram, for if we set out along the line *OX* distances through which the pressure works, and along *OY* the corresponding pressures, then we shall have a figure traced, the area of which is the product of the pressure into the distance through which it acts. An important and interesting application of the indicator diagram was the determination by Joule of the work done in compressing a quantity of air.

The work in one of his experiments (see page 103) was equal to compressing 21·634 cubic feet of air into the space of 1 cubic foot in a tube, of which the area was 11·376 square inches. The original pressure on the 21·634 cubic feet was 168·5 lb. on an area of 11·376 square inches, and gradually rose, according to the law of Boyle, until the final pressure was 3648·7 lb. on 11·376 square inches.

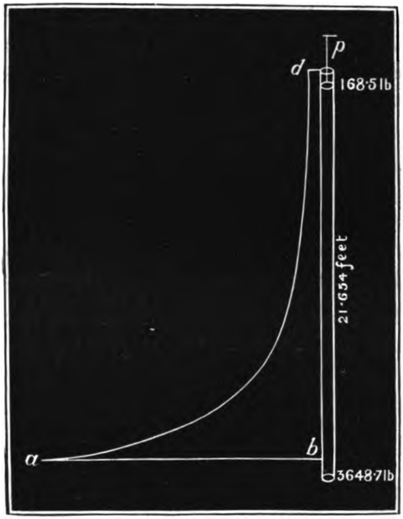

Fig. 66.

A diagram was set out, as in fig. 66, marking the rise in pressure as the air was compressed, and the area *abd* was a measure of the work done. The curve is the rectangular hyperbola, and the area of the figure *abd* is given by the expression

ab × Naperian log of *bd*.

Now the Naperian logarithm is the same as 2·302585 × common logarithm; hence *s*, the work done in foot-pounds, was given by the expression

$$s = 3648{\cdot}7 \times 2{\cdot}302585 \times \log 21{\cdot}654 = 11{,}220{\cdot}2$$

Illustrations of this method are given in Lesson XIII.

Index Notation. In many of the problems given in the body of the work the numbers contain so many digits, or the decimal fractions so many cyphers, that it becomes troublesome to numerate them. In index notation usually one digit only is written, followed by decimals, and this is multiplied by the power of 10, which will give the required number. Thus, 123456700 would be expressed as $1{\cdot}234567 \times 10^{8}$, while 0·0004873 would be expressed as $4{\cdot}873 \times 10^{-4}$.

If the student uses logarithms, and he cannot be too strongly advised to do so, he will note that the index 8, or $\bar{4}$ in the above example, shows the characteristic of the logarithm. I have prepared a set of logarithm tables (particulars of which are given at end of the book) which will be found convenient in working the exercises.

When indices are given in a fractional form, such, for example. as $a^{\frac{3}{4}}$, it is meant that the quantity a is to be raised to the power of 3, and the 4th root of this quantity taken, that is, the expression is the same as $\sqrt[4]{a^{3}}$. Again, $v_1^{1{\cdot}408}$ means the 1·408th power of v_1, or, what is the same thing, $v_1^{\frac{1408}{1000}}$, which is the 1000th root of v_1 raised to the 1408th power. We have in Lesson XIII., page 114, the following statement :—

$$\left(\frac{p_1}{p_2}\right)^{\frac{1}{1{\cdot}408}} . \frac{p_2}{p_1} = \left(\frac{p_2}{p_1}\right)^{\frac{0{\cdot}408}{1{\cdot}408}}$$

For $$\left(\frac{p_1}{p_2}\right)^{\frac{1}{1{\cdot}408}} . \frac{p_2}{p_1} = \frac{p_1^{\frac{1}{1{\cdot}408}}}{p_1} \times \frac{p_2}{p_2^{\frac{1}{1{\cdot}408}}}$$

But $$\frac{p_1^{\frac{1}{1{\cdot}408}}}{p_1} = p_1^{\frac{1}{1{\cdot}408} - 1} = p_1^{-\frac{0{\cdot}408}{1{\cdot}408}} = \frac{1}{p_1^{\frac{0{\cdot}408}{1{\cdot}408}}} \qquad (a)$$

bearing in mind that $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$

and $$\frac{p_2}{p_2^{\frac{1}{1\cdot408}}} = p_2^{1 - \frac{1}{1\cdot408}} = p_2^{\frac{0\cdot408}{1\cdot408}} \qquad (b)$$

Hence, from (*a*) and (*b*) we have

$$\frac{1}{p_1^{\frac{0\cdot408}{1\cdot408}}} \times p_2^{\frac{0\cdot408}{1\cdot408}} = \left(\frac{p_2}{p_1}\right)^{\frac{0\cdot408}{1\cdot408}}$$

REFERENCE CHAPTER B.

Absolute Units.

Two systems of absolute units are in common use in this country among scientific men: one is known as the foot. pound, second system, and might be expressed as the *F.P.S.* system; the other is the centimetre, gramme, second system, known as the *C.G.S.* system. These units are each often varied to suit particular measurements. Thus, in magnetic measurements, grain is substituted for the pound in the English system. In the metric system the millimetre (barometer measurements), the litre (density of gases), &c., are used.

The following table shows the principal units in the two systems :—

Unit of	*F.P.S.* system.	*C.G.S.* system.
Length.	*The foot.* One-third of the standard yard, preserved by the Board of Trade.	*The centimetre.* One-hundredth of the standard metre, preserved in Paris.
Mass.	*The pound.* A mass equal to that of the standard pound, preserved by the Board of Trade.	*The gramme.* A mass equal to that of one cubic centimetre of water at 4°C.
Time.	*The second.* The 1/86400th part of a mean solar day.	*The second.*

Unit of	*F.P.S.* system.	*C.G.S.* system.
Force.	*Poundal.* That force which can give a velocity of one foot per second, while acting for one second, upon a mass of one pound.	*Dyne.* That force which can give a velocity of one centimetre per second, while acting for one second, upon a mass of one gramme.
Work.	*Foot-poundal.* Moving a mass of one pound, through a distance of one foot, against a force of one poundal.*	*Erg.* Moving a mass of one gramme, through a distance of one centimetre, against a force of one dyne.
Rate of working.	*One foot-poundal per second.*	*One erg per second.*

The density or *volume mass*† of a substance in the F.P.S. system is the mass of a cubic foot of it, *e.g.*, water 62·5 lb. (nearly), dry air (at 30 in. mercurial pressure and temperature of 62°F.) at the latitude of London = 0·076315‡ lb. In the C.G.S. system density of a substance is the mass of a cubic centimetre of it, *e.g.*, water one gramme, air at 0°C. and 76 cm. = 0·001293934 gramme.§

When a formula has been obtained, then in applying the formula it is immaterial what units are adopted, but to obtain a possible result the units must be rigidly adhered to. For example:—

---

* The *foot-pound* must not be confused with the foot-pound*al*. The foot-pound is not an absolute unit, for it varies with the place where the work has to be performed. At Greenwich, where $g=32{\cdot}2$ nearly, the foot-pound will be 32·2 poundals. At the pole it will be greater, at the equator less.

† An expressive term introduced, I believe, by Dr. Hofmann as volume weight.

‡ *Nature*, June 30th, 1887, page 205.

§ *Report Weights and Measures Act*, 1883, page 16.

As reasoned out in Lesson II., page 15, the number of vibrations of a string fixed at both ends sounding its fundamental is shown to be

$$n=\frac{1}{2l}\sqrt{\frac{t}{m}}$$

when n represents number of vibrations per second.

l represents the length of the string.

t represents tension of the string.

m represents the mass of unit length of string.

Suppose we take the question given at a recent examination in the advanced stage of Sound, Light, and Heat, which was:

"If a string 24 inches long weighs half an ounce, and is stretched with a weight of 81 lb., find its rate of vibration when bowed or struck."

We use the F.P.S. system in solving this, and we have

$l=2$ *feet.*

$t=81\times 32$ poundals, g taken as 32.

$m=\frac{1}{4}$ oz. $=\frac{1}{64}$th lb.

$$\text{Hence } n=\frac{1}{2\times 2}\times\sqrt{\frac{81\times 32}{\frac{1}{64}}}=\tfrac{1}{4}\times 9\times 32\times\sqrt{2}=72\times 1{\cdot}414$$
$$=101{\cdot}8 \text{ nearly.}$$

An application of the same formula was given in the Honours Examination of 1885, but in which the C.G.S. system was used. The question was:

"A metal wire 120 cm. long, and weighing 1 kilogramme per 600 metres, is stretched by a weight of 10 kilogrammes. The wire is found when plucked to perform 100 complete vibrations per second. Determine from these data the value of the acceleration of gravity."

Here in applying the formula we have

$n=100.$

$l=120$ cm.

$t=10{,}000\times g$ dynes.

$m=1000/60000$, or $1/60$ gm.; so that

$$100=\frac{1}{2\times 120}\sqrt{\frac{10{,}000g}{\frac{1}{60}}}\text{, from which}$$

$$10{,}000\times 60g=(100\times 2\times 120)^2\text{, and } g=960 \text{ dynes.}$$

Besides the two systems of units mentioned, others are sometimes used. In magnetic observations, the foot, *grain*, second system is used in this country. To convert the numerical result of an expression in one system into the numerical result as expressed in another, care has to be exercised in reference to what are called "dimensions" of the different units. The mode of changing one system of units to another is given in *Arithmetical Physics*, *Part II*B.

REFERENCE CHAPTER C.

General Phenomena of Wave Interference.

Wave interference explains so many phenomena in experimental science that the student may with advantage study this subject very carefully. To begin with, it is assumed that the student is familiar with Lesson I, "Waves in General," pages 1-4, of Part I. of this work.

Although the principles about to be explained are applicable to all kinds of waves, it will be most convenient to consider only waves produced by particles vibrating in one plane; for example, such waves as those formed by a string compelled to vibrate in one plane. The wave *C* of fig. 3 of Part I. is a wave of this kind.

Plate 1. Let *abc* ...... *ghk*, *A* of fig. 1, plate I., represent a wave at a particular moment. Draw the neutral line *ak*, and at every eighth length of the wave put in arrows, z_0, z_1, z_2, z_3, z_0, $\bar{z}_1$, $\bar{z}_2$, $\bar{z}_3$, z_0. Then, while *abc* ...... *ghk* represent the form of the wave at the moment, the arrows z_0, z_1, z_2, &c., represent the potential energies of the particles at that moment. If the wave is moving from left to right, then, when it has moved through one-eighth of a wave, the potential energies of the particles will be as represented at *B* of fig. 1, plate I., and when it has moved through one-quarter of a wave, they will be as represented at *C* of the same figure; while, if the wave is moving from right to left, then, when it has moved one-eighth of a wave length in this direction, the potential energies will be represented at *D*; and when it has moved one-quarter of a wave from right to left, they will be as at *E* of the same figure. It will be noticed, from the above remarks, that the effect of a progressive wave may be studied by considering the wave form *abc* ...... *ghk* to move bodily in one direction or the other. This mode of viewing the progress of a wave simplifies the subject in many cases.

To find the form of wave resulting from two waves encountering each other, both waves travelling with the same velocity and in the same direction. Let *A* and *B*, fig. 2, plate I., represent two waves in the position they are at a particular moment. Draw the neutral lines, *pq*, *rs*, and also any convenient number of vertical lines passing through both waves.* Now, from the straight line, *tu*, in lower part of figure, set off distances above and below, corresponding to the algebraic sum† of the arrows, which are on the same verticals immediately above, and we shall have the form of the new wave, as shown at *C* of fig. 2. Of course, this represents the form of the wave at the particular moment considered; but as *A* and *B* are moving with the same velocity, and in the same direction, the compound wave will always have the same form, but it will advance just as the components advance. Hence the wave *C* moving forward is the resultant of the two waves *A* and *B* moving forward together. **Plate I.**

It may be mentioned here that the elementary wave forms, *A* and *B* of fig. 2, are called simple harmonic waves, for they cannot be resolved into simpler waves. On the other hand, the wave *C* is a composite wave, made up of the two simple waves *A* and *B*.

In fig. 2 the resultant wave *C* is not complete. The student should set out on "section paper" the waves *A* and *B*, and continue them until both waves return to the neutral line together. He will find that this will be the case when he has drawn three of the *A* waves and four of the *B* waves.

An important application of this method is the case of two waves, of equal lengths and amplitudes, interfering with each other, as in fig. 3, plate I., where A_1 and B_1 represent two waves, one of which, B_1, is started just a wave length in advance of the other; it will be found, by the method just described, that the resultant wave C_1 is one of double amplitude. Again, in the same figure, A_2, B_2, are two waves of equal amplitude; but in this case B_2 is started half a wave length in advance of A_2, and it will be seen that the resultant wave C_2 is a straight line, that is, the two waves destroy each other.

---

* These vertical lines, which were originally drawn where the arrows occur, are not represented in the plate for the sake of clearness. Convenient "section paper," suitable for drawings of the kind described, can be bought at the Midland Educational Co., Corporation Street, Birmingham, and, no doubt, elsewhere, at one penny per sheet, or one shilling per quire.

† In other words, where the parts of the component waves are both above or both below the neutral line add the verticals together, but where one vertical is above and the other below, then take the difference, and mark it off in the direction of the greater. For example, the line *dx* of the *C* wave $= d_1x_1 + d_2x_2$, but *rx* of the *C* wave $= -r_1x_3 + r_2x_4$.

In the two examples above it has been supposed that there is a difference of *one* wave, or *one-half* wave, respectively, but the student will easily see that any multiple of whole wave difference will produce the same effect as if there were only a single wave difference, and any odd multiple of a half-wave length difference will produce the same effect as though there were only one half-wave length difference. So that we may say,

If two waves of equal amplitudes and lengths are propagated simultaneously in the same direction, and with the same velocity, then, if the difference of phase of the two waves be nought, two, four, or any even number of half-wave lengths, the resultant wave will be one of double intensity; but if the difference of phase be one, three, five, or any odd number of half-wave lengths, then the resultant wave will be nil, that is, the two waves will destroy each other where interference takes place.

If the two waves referred to above have different amplitudes, but the same lengths, then, if the difference of phase be an even number of half-wave lengths, the amplitude will be the sum of the amplitudes of the two waves; but if the difference of phase be one, three, five, or any odd number of half-wave lengths, the resultant wave will have an amplitude equal to the difference of that of the two waves.

If the two waves have different lengths, and either the same or different amplitudes, the resultant wave will rise and fall in intensity periodically, as will be seen on trial. This point has already been referred to under the head of beats, see page 25.

So far reference has only been made to two simple harmonic waves, and the mode of finding their resultant wave. But the same method may be applied to finding the resultant of three or more simple harmonic waves propagated simultaneously. Although difficult to do, it can be seen that any arbitrary periodic curve might, in imagination, be resolved into a series of simple harmonic waves. In fact, this is Fourier's theorem, already referred to on page 14.

Composite waves may often be seen recorded on the piers of a bridge. For example, underneath one of the arches of Westminster Bridge I have often observed the trace of a wave-form left on the pier. I am indebted to a friend, Mr. Alyn Williams, for some photographs of the marks on Westminster Bridge, from which the sketch of fig. 5, plate 2, has been taken. The irregular line of fig. 5 has been produced, perhaps, by a great wave of a passing steamer, smaller waves from the action of the wind, waves produced by the oars of a rowing boat, &c., and all have conspired to trace the form depicted. Given sufficient skill, it would be possible from this composite wave to say what particular disturbances have produced the form depicted. Plate 2.

Waves moving in opposite directions. Interference of direct and reflected waves. Let A_1, B_1, fig. 4, plate I., represent two waves, of equal length and amplitude, moving in opposite directions, as indicated by the arrows. At one moment they will be in the position represented at A_1 and B_1 of fig. 4, and the resultant effect *at that moment* will be the wave line of double intensity, as represented at C_1. Let, now, each wave move one-fourth of a wave forward, and we have the waves as represented at A_2, B_2, and the resultant at this moment is the form C_2, that is, it is a straight line. When the waves have each moved through another quarter wave, as at A_3, B_3, the resultant at this moment will be the form C_3; another quarter forward they will again give a straight line, so that the effect of two waves, of equal length and amplitude, encountering each other is to produce, not a progressive wave at all, but (if the waves are propagated on a cord) the cord will be divided into the well-known condition of nodes and ventral segments. Plate 1.

In fig. 4, plate 1, only three positions of interference of the two waves have been given. The student should work out on his drawing board or squared paper the form of the resultant wave for, say, each sixteenth of the motion forward of the one wave, and backward with the other, when he

Plate 1. will get the series of forms of fig. 5, plate I., the successive forms being indicated by continuous and dotted lines.

It may be noticed here, as a help towards understanding the following paragraph, that the position of the nodes in the stationary waves is found by drawing vertical lines through the neutral points of both waves when they are in coincidence, as at A_1, B_1, of fig. 4, plate 1, or through the crest of the one and trough of the other wave when they are in opposition, as at A_2, B_2, of the same figure.

The production of nodes and ventral segments by opposite motions of equal waves being understood, their formation from direct and reflected waves naturally follows.

When a wave falls upon a reflecting surface, the way in which it is reflected will depend on circumstances. With a cord attached to a support, with a water wave striking a wall, with a wave in air striking the *closed* end of an organ pipe, and generally a wave in a rare medium falling upon a dense medium, as when light waves pass from air to glass, the motion of each particle is reversed. Thus the waves *a b c* ...... *g h k* in upper part of fig. 67 striking the dense obstacle *O* has the motion of each particle reversed, and this reversal gives rise to the reflected waves represented immediately below. By the time the motion of *a* of the direct waves reaches *O* the reversed motion of *k* will be transmitted to *k′*, when *b* reaches *O* the reversed motion of *h* will be transmitted to *h′*, by the time *c* reaches *O*, *g* will be at *g′*, and so on. Thus the direct waves of *D* give rise to the reflected waves of *R*, and as those of *D* travel directly towards, and those of *R* directly from, the obstacle, and both travel with the same speed, their joint effect is the stationary wave represented at *S*, with a node at *O*, and other nodes at every half-wave length.

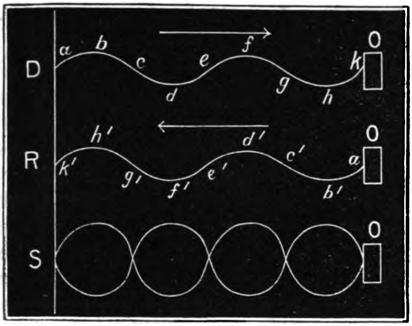

Fig. 67.

When a wave in an organ pipe strikes the *open* end of the pipe, when a wave in a brass rod reaches the free extremity of the rod, when an ether wave in glass meets a surface of water, or generally when a wave in a dense medium meets the surface of a rare medium, a reflected wave is produced; but here the several parts of the wave are reflected without inversion of their motion. We may call such a surface a negative reflecting surface.

Let the waves *a b c* ...... *g h k* of *D*, fig. 68, strike against a negative reflecting surface *O*. By the time the motion of *a* reaches *O* the motion (direct) of *k* is transferred to *k'*, by the time *b* reaches *O* the motion (direct) of *h* is transferred to *h'*, and so on; so that by the time the motion of *a* has reached the reflecting surface, the reflected waves *R* have been formed, and we have now simply to consider the result of the direct waves *D* interfering with the reflected waves *R*; and by the principles already explained, the resultant will be a series of stationary waves *S*, in which a ventral segment is at the reflecting surface, and a node one-fourth of a wave length from the surface.

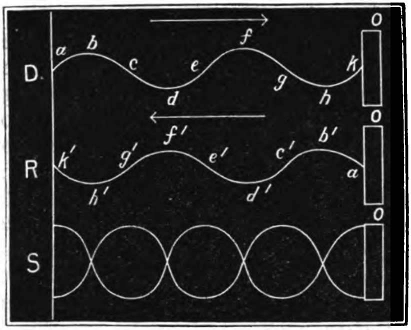

Fig. 68.

The stationary waves represented at *S*, fig. 68, correspond—as the student will, no doubt, notice—to the stationary waves of an open organ pipe, the negative reflecting surface *O* corresponding to the air at one of the open ends of the pipe. On the other hand, *S* of fig. 67 corresponds to stationary waves in a closed organ pipe, *O* corresponding to the closed end of the pipe.

Interference of two plane waves at right angles to each other. Suppose we have two plane waves, the planes of which are at right angles to the plane of the paper, and to each other. Then each particle of each wave vibrates in a plane parallel to that

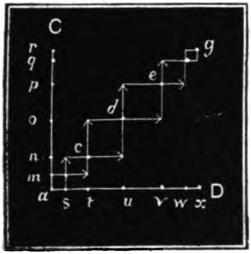

Fig. 69.

of the paper. Let *C*, fig. 69, represent the trace of one wave, and *D* the trace of the other; while *a m n o*...... are successive positions of particles in *C*, and *a s t u* ...... successive positions of particles in *D*. We have now to find the form of the wave resulting from the interference of the two waves, that is, the form of the wave which will be produced in the medium on the other side of the paper.

Suppose the two waves applied to each other so that the phase of each is the same, as represented in fig. 69. The particle *a* is solicited by two forces, one of which, if acting alone, would carry it to *m*, while the other would carry it to *s* in the same time. It is obvious that it will be carried to the point *b* along the diagonal of the parallelogram *a m*, *a s*; here it will be solicited by the two forces *s t* and *m n*, applied at *b*, and these will carry it to *c*. Similarly, the forces *t u* and *n o* will carry it to *d*; and, continuing the same reasoning, it is evident that the particle will vibrate between *a* and *g* in a straight line at an angle of 45° to either of the components, and that the extent of this resultant vibration is as many times greater than its components, as *a g* is greater than *a r* or *a x*. Since each particle below the paper will be acted upon in succession in the same manner, the resultant wave will be *plane*, and it will be one of which the amplitude is *a g*, and its plane inclined at 45° to either of the components.

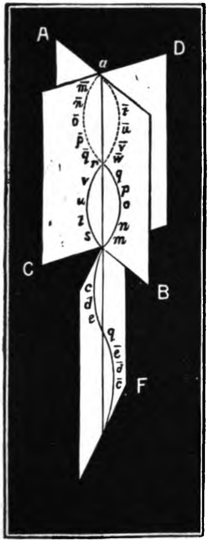

Fig. 70.

If the component wave *D* has a different amplitude from the component *C*, the wave will still be a plane one; but the angle of inclination will be different.

In fig. 70 is an attempt to depict this conversion of two plane waves at right angles to each other into a plane wave intermediate between them.

Take another case. Let the components still be at right angles to each other, but instead of the components being in the same phase, let one component be a quarter of an undulation behind the other. Then the particle *a*, fig. 71, will be solicited by the component *D* with a force represented by *a v* at the same time that it is solicited by the component *C* with a force represented by *a m*. These forces will carry it to *b*, at the extremity of the diagonal of *a v* and *a m*; from *b* it will be carried to *c* from the action of the forces *v w* and *m n*, and so on. Carrying this method out, it will be found that the resultant vibration is a circular motion, as represented by *a b c* ......, and as each particle in succession takes up this motion, the resultant wave becomes a "circular" one. A corkscrew, if turned, is a representation of such a wave.

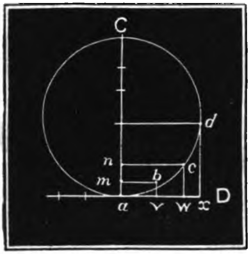

Fig. 71.

An attempt to depict the conversion of two plane waves at right angles to each other is represented in fig. 72. To give an idea of the circular wave produced, it is represented as moving round a cylinder. An apparatus, constructed by the author to illustrate the interference of two plane waves at right angles to each other, will be found in the *Philosophical Magazine* for 1881, page 145.

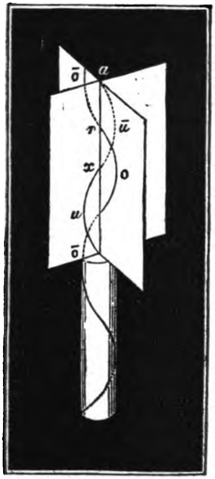

Fig. 72.

Just as two component waves can be formed into one resultant wave, so a single wave may be resolved into two components. Thus the wave *a g*, fig. 69, may be resolved into the two waves *C* and *D*. Similarly, a circular wave may be resolved into two others, at right angles, differing by a fourth of an undulation. This analysis and composition of waves is the key to the explanation of the varied and intricate phenomena of polarized light.

Principle of secondary waves. Explanation of reflection and refraction on the undulatory hypothesis. If a small portion of water be set vibrating so that the motion of the particles is in a vertical plane, the motion is propagated in all directions, and we have the familiar wave surface, such as is produced by throwing a stone into still water. The principle of secondary waves declares that the same effect would be produced whether we imagine the grand or primary wave to be produced by a single motion from its centre or from a multitude of minor or secondary waves, each starting from an infinity of points scattered over its surface, the particular phases of these secondary waves being coincident with that of the primary.

Plate 2. Fig. 3, plate 2, is a plan of the surface of a pond in which a wave is being propagated from a centre C. The circles a, b, c, represent the most elevated points (or crests) of the waves. Suppose the wave has reached c, then after a short interval of time the crest d will be formed. To understand how this crest will be formed on the principle of secondary waves, suppose a multitude of secondary waves to start out from the circle c at the same moment. The crest of each of these secondary waves will be small circles with a radius equal to ab or bc, as represented, and it is obvious that a portion of the crest of each of the secondary waves will reach the circle d at the same moment, so that as regards the effect of these secondary waves in the direction of the normal to the wave, such as Cn, Cn', it coincides with what results from the grand or primary wave. It remains, however, to be considered what will be the effect of the secondary waves setting out from c_1, c_2, &c., in any other direction than that of the normal, such as c_1n, c_2n, cn', c_1n', and it can be shown that in these directions the secondary waves practically destroy each other.

Destruction of secondary waves except in direction of primary. With a pair of compasses strike from centre C, fig. 1, plate 2, a series of circles with a constant difference in radii, thus representing a series of waves propagated from the centre C. Take any point X, and join XC. Note that where the circles cut this normal as at a, b, c, d, ... the distances ab, bc, ... will represent wave lengths, while points such as v_1, v_2, v_3, half way between ab, bc, ... enable us to measure half-wave lengths.

Now set one leg of the compasses at X, and extend the other to the point v_1, and with this radius cut the circle ap at the point 1, and draw the dotted line $X1$. Now with radius Xb cut the circle ap at l, and draw the continuous line Xl; again with radius Xv_2 find out position of the point 2, and draw the dotted line $X2$, and so go on, as shown in the figure. Let us now see to what this construction leads.

The molecules of water about the point 1 are propagating waves at the same time that the molecules about the point l are propagating waves; but it is clear that as there is a difference of distance between X_1 and Xl of half a wave length, some portion of the secondary wave from 1 will be destroyed by those from l; the same holds in reference to secondary waves propagated from points in the neighbourhood of 2 and m, 3 and n, and so on. Now, as the lines $X1$, Xl, $X2$, Xm, &c., become more oblique, or as X gets nearer to a, the interference is more and more complete, so that within a moderate distance of a we may regard the secondary waves as neutralising each other, and the primary wave alone acting. (See *Encyclopædia Metropolitana*, article *Light*, page 461; Herschell's *Popular Lectures on Scientific Subjects*, Strahan, 1873, page 329; Lloyd, *Wave Theory of Light*, Longmans, 1857, page 70.)

It would seem from this reasoning that if we were to stop out portions of the primary wave about the points 1, 2, 3, ... that the total disturbance at X should be increased. "This startling conclusion is confirmed by experiment." Lloyd, *Wave Theory of Light*, 1857, page 71.

The effect of a series of partial waves in producing a grand wave is well illustrated when a stick is drawn through water, or when a boat is moving along a canal. As the stick or boat moves forward, waves start out from each point of its path, and just as a wave is beginning to start from the prow of the boat, others, which have been started a short time previously, have spread out greater and greater distances, according to the time which has elapsed since their commencement. The result of this series of waves is the grand wave, with which everyone is familiar, spreading from the boat in the shape of the letter *V*.

It is pretty obvious that the faster the boat is towed the more acute is the angle made by the two arms of the *V*. If the arms of the *V* enclose 90°, that is, 45° on each side of the median line, then the speed of the boat will be equal to the rate of propagation of waves in water.

Having given a series of Secondary Waves to find the Resultant Primary Wave. Suppose a number of secondary waves have been propagated at different intervals of time from the centres *a*, *b*, *c*, *d*, fig. 2, plate 2, and intermediate points, it is required to determine the form, that is, the wave front, of the grand wave produced by these secondary waves. From the points *a*, *b*, *c*, *d*, &c., draw circles with radii of lengths corresponding to the length of time during which each wave has been in existence, and then draw the thick line *ABCD*, touching all the circles of these secondary waves, this line is the wave front. The direction of propagation of any part of the wave, that is, the normal to the wave front at any point, is found by drawing a radius of the secondary wave which touches the wave front at that point. Thus, *aA*, *bB*, *cC*, *dD*, are normals to wave front *ABCD*, and indicate the direction of propagation of the wave at the points *ABCD*.

Plate 2.

Explanation of reflection and refraction on the undulatory theory. Let *ABCD*, fig. 4, plate 2, represent an elevation of a brick-like mass of glass or other transparent body, having an index of refraction = 2. *tadw* represents a beam of light falling from vacuo upon the upper surface of the mass of glass. Draw *ap* at right angles to the beam, hence corresponding to the wave front. The moment the portion *a* of the beam strikes the glass, two secondary waves will set out from that point: one outwards into vacuo with the same velocity as the incident light, the other inwards in the glass with half this velocity (since $\mu = 2$), so that by the time the portion of the wave front *p* reaches the glass at *d* the secondary wave from *a* will have reached a distance ap' in vacuo, equal to *pd*, while in the glass it will have reached a distance ap_2, equal to one-half of *pd*. Similarly when *q* reaches the glass at *d* the secondary wave in vacuo will have reached a distance $bq' = qd$, and the secondary wave in the glass will have reached a distance $bq_2 = \frac{1}{2}qd$; similarly $cr' = rd$, and $cr_2 = \frac{1}{2}rd$. So that when the portion of the wave front p reaches d a multitude of secondary waves setting out from the line ad will have been formed, of which a few have been indicated. These secondary waves will together produce a reflected wave, of which $p'd$ (touching all the circles) is the wave front, and a refracted wave of which p_2d (touching all the circles) is the wave front. The lines ap', bq', &c., at right angles to the wave front of the reflected wave give the direction of propagation of the reflected wave, while the lines ap_2, bq_2, &c., give the direction of propagation of the refracted wave. It remains now to show that the angle of reflection Nat' is equal to the angle of incidence Nat, and that the sine of the angle of refraction $N'ap_2$ is one-half of the sine of the angle of incidence Nat.

In the right-angled triangle, apd, the angle pad is 90° less the angle pda, that is, 90° less the angle taA, that is, it is equal to the angle taN, the angle of incidence. Again, in the right-angled triangle $ap'd$, the angle $p'da$ is 90° less the angle $p'ad$, that is, it is equal to the angle Nat', which is the angle of reflection. So far, then, we show that $pad =$ angle of incidence, and $p'da =$ angle of reflection. Now, with the two triangles apd and $ap'd$, they are on the same base, they have equal angles at p and p' (right angles), and have their sides dp and ap' equal; hence it follows that they are equal, and that the angle pad is equal to the angle $p'da$, that is, that the angles of incidence and reflection are equal to each other.

Again, as $pad =$ angle of incidence, therefore,

$$\frac{pd}{ad} = \text{sine of angle of incidence.} \qquad \text{(i.)}$$

also $adp_2 =$ angle of refraction, for it is equal to the angle xap_2 (since both angles $= 90° - p_2ad$), so that

$$\frac{ap_2}{ad} = \text{sine of angle of refraction.} \qquad \text{(ii.)}$$

$$\text{Hence (i.} \div \text{ii.)} \quad \frac{pd}{ap_2} = \frac{\text{sine of angle of incidence}}{\text{sine of angle of refraction}} = \mu = 2$$

Hence the law of sines follows, and we see that the index of refraction on the undulatory hypothesis merely expresses the ratio of the velocity of propagation in the two media.

APPENDIX
OF
REGENT EXAMINATION PAPERS
ON
SOUND, LIGHT, AND HEAT,
WITH THE QUESTIONS CLASSIFIED.

SERIES I.

SCIENCE AND ART DEPARTMENT QUESTIONS,
ADVANCED AND HONOURS STAGES,

1885-90.

NOTE.—The questions are classified under the headings of the syllabus, the heading for each division being indicated by quotation marks. Italics refer to the advanced stage, underlined italics to the honours stage. The number of the question and year in which it was given are indicated by figures in brackets at the end of each question.

The letter H means that questions to which it is appended are from the Honours papers. All other questions are from the advanced papers.

SUB-SUBJECT I.—SOUND.

A.—"Cause of sound. Vibrations of bodies. Propagation of sound through air. Vibrations of air particles, waves, wave-length."

"The construction and use of the air-pump. Boyle's law. Meaning of the term elasticity as applied to gases."

1. How can Boyle's law for gases be verified by direct experiment? (21, 1888.)

B.—"Velocity of sound in air. The effect of changes of elasticity and density on the velocity. Effect of changes of temperature on the elasticity and density of gases."

"Laplace's correction of Newton's formula."

"Influence of fog, rain, &c., on the velocity of sound in air."

"Velocity of sound in liquids and solids."

"Methods of measuring velocity of sound in gases, liquids, and solids."

"Reflection of sound. Echoes."

1. Explain how to determine experimentally the velocity of sound in hydrogen. (24, 1886.)

2. How would you compare the velocities of sound in different gases? (21, 1887.)

3. Define "elasticity" and "density," and explain how the velocity of sound depends upon them. (22, 1888.)

4. What effect does pumping half the air out of a closed vessel produce on the velocity of sound passing through it? Also what is the effect of compressing the air? (The temperature in both cases is supposed to be constant.) (23, 1888.)

5. Explain and illustrate the mode of vibration of air. Taking 1120 feet per second as the velocity of sound in air, find the number of vibrations which a middle C tuning fork giving 256 vibrations per second will make before its sound is audible at a distance of 154 feet. (21, 1889.)

6. How could you show by experiment that sound travels more quickly through warm than through cold air? What law connects velocity of sound in air or gas with temperature? (21, 1890.)

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7. Distinguish between the isothermal and the adiabatic elasticity of a gas, and show their relation to each other Hence explain how the velocity of sound in the gas may be calculated. In the case of atmospheric air, point out what data are necessary for the calculation. (H. 45, 1887.)

8. The specific gravity of a certain gas under a pressure of 75 centimetres of mercury and at 0°C. is one-thousandth that of water. What is the velocity of sound in it at that temperature? What is it also at the temperature of 100°? Examine whether altering the pressure on the gas will affect the velocity.
(Take the ratio of the two specific heats as 1·4.) (H. 41, 1888.)

C.—"Transverse vibrations of strings. Nodes and ventral segments. Change of pitch due to changes in the length, diameter, material, and stretching force of a string vibrating transversely."

"*Longitudinal vibrations of strings. Change of pitch due to length, diameter, material, and stretching force of a string vibrating longitudinally.*"

"*Longitudinal vibrations of rods.*"

1. A string is stretched on a sounding board, and is rubbed with a resined glove in the direction of its length. Explain what effect will be produced on the musical note by tightening the string while it is vibrating. (25, 1885.)

2. Four strings of the same length and materials, but of different thicknesses, are stretched on a violin, and tuned so as to give successive fifths. If the tension be the same, compare the thicknesses of the strings. (21, 1886.)

3. In the last question what must be the tension on the several strings that they may all give the same note? (22, 1886.)

4. Four strings of the same material and length, but of thicknesses 1, 1·5, 2·5, and 3, are stretched on a violin, and tuned so as to give successive fifths. Compare the tensions of the several strings. (The thinnest string is to give the highest note.) (23, 1887.)

5. If a string 24 inches long weighs half an ounce, and is stretched with a weight of 81 lbs., find its rate of vibration when bowed or struck. (22, 1889.)

6. State the law regulating the transverse vibration of strings under tension? What data are needed to determine the velocity of transmission of transverse pulses along a flexible cord? (22, 1890.)

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7. A metal wire 120 cm. long, and weighing 1 kilogramme per 600 metres, is stretched by a weight of 10 kilogrammes. The wire is found when plucked to perform 100 complete vibrations per second. Determine from these data the value of the acceleration due to gravity. (H. 41, 1885.)

D.—"*Transverse vibrations of rods and plates. Tuning forks. Bells.*"

1. State the laws of vibrations of rods fixed at one end. What relation holds between the numbers of vibrations for the successive overtones in such rods? (23, 1886.)

2. State the modes of vibration possible to bells, and how you would illustrate them. (24, 1889.)

E.—"*Methods of determining the pitch of musical sounds.*"
"*The siren.*"
"*Interference of sound waves. Beats.*"
"*Harmonics.*"

1. How would you show experimentally that the tone of a stretched string vibrating transversely is compound in its nature? If you wished to suppress the second harmonic (the twelfth above the fundamental), where would you pluck or strike the string? (23, 1885.)

2. What conditions are necessary in order that two sound waves reaching the air by different paths may produce silence? (25, 1886.)

3. Explain how to determine the time of vibration of a given tuning fork, and state what apparatus you would require for the purpose. (24, 1887.)

4. Describe an experiment to prove that two sounds may produce perfect silence. (25, 1887.)

5. Explain the occurrence of "beats" when two notes nearly of the same pitch are sounded together. How comes it that open organ pipes nearly an octave apart give beats? How comes it that feeble beats are heard even from two stopped pipes, or two tuning forks, when nearly an octave apart in pitch? (24, 1888.)

6. Given a set of standard tuning forks, how can you determine accurately the rate of vibration of any sounding body? (23, 1889.)

7. Explain how to determine the time of vibration of a given tuning fork, and state what apparatus you would require for the purpose. (24, 1890.)

F.—"*Musical scales. Temperament.*"
"*Principles of Harmony.*"

1. Point out the differences between the diatonic scale and the scale of equal temperament. Why is the latter employed in the piano and organ? (22, 1885.)

2. Give the relations of the number of vibrations of the successive notes of the diatonic scale and the Pythagorean scale, and explain for what purpose each scale is suitable, and why? (25, 1888.)

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3. Give the number of vibrations corresponding to each note of the diatonic scale ($c' = 264$). Point out clearly the advantages and disadvantages of the equally-tempered scale. What other scales have been used or proposed? (H. 41, 1886.)

G.—"Vibrations of columns of air. Mouth (or flue) organ pipes open and stopped. Other notes besides the fundamental yielded by organ pipes. *Reed organ pipes.*"

1. Explain why an organ pipe which is tuned at one temperature is out of tune at a different temperature. What data would you require to enable you to calculate the change of pitch of an organ pipe due to a rise of temperature from 0°C. to 20°C.? (21, 1885.)

2. Describe the method of testing the state of disturbance of the air at any part of an open organ pipe when a musical note is being produced from it.

What results will be obtained when the first harmonic is being produced from the pipe? (22, 1887.)

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3. State and explain the effect of immersing an organ pipe in hydrogen and blowing it with hydrogen instead of as usual with common air.

If the tube were an open one 4 feet long, about how many vibrations per second would you expect the note emitted to have in each case?

State precisely how the tone of an organ pipe varies with temperature. (H. 41, 1890.)

H.—"Loudness, pitch, and quality of musical sounds." "*Resonance. Resonators. Method of using resonators to distinguish the different qualities of compound notes.*"

1. When a tuning fork is set in vibration, and held close to one end of a glass tube 20 inches long, and open at both ends, an augmentation of sound takes place. If the tube is longer or shorter than 20 inches, the increase of sound is not so great. How do you explain these facts? and how could you calculate from them the pitch of the tuning fork? (24, 1885.)

2. What are sympathetic vibrations? Account for them, and explain under what circumstances they may be produced. (23, 1890.)

ANSWERS.

B. 5. 35·2. **B. 8.** 37,428 cm.

C. 2. 1 : 2/3 : 4/9 : 8/27.

C. 3. 1 : 4/9 : 16/81 : 64/729.

C. 4. 64 : 100 : 81 : 81.

C. 5. 101·8. **C. 7.** 960.

G. 3. Taking density of air as 14·44 times that of hydrogen, the velocity of sound in hydrogen is 3·8 times that of air; hence, assuming velocity in air as 1120 feet, the vibrations will be 140 and 532.

SUB-SUBJECT II.—LIGHT.

K.—" Propagation in straight lines. Shadows. Penumbræ. Eclipses. Velocity of light. Modes of determining it. Intensity of light. Law of change of intensity with change of distance. Photometry."

1. Explain the principle of the Bunsen (or grease-spot) photometer. How would you prove that the illumination of any surface is inversely as the square of its distance from the source of light? (26, 1885.)

2. Describe the observations which led Rœmer to the conclusion that light does not travel with infinite velocity; and explain how he computed the actual velocity. (28, 1885.)

3. Rœmer determined the velocity of light by observations of the eclipses of one of Jupiter's moons. State clearly the nature of his observations, and the reasoning by which the velocity of light was deduced from them. (27, 1886.)

4. Describe the method adopted by Fizean for determining the velocity of light. (28, 1887.)

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5. In comparing the intensities of different sources of light, show that a steady, broad flame with an open slit in front of it may be used as a standard of light provided the measurements are made from the slit. What precautions are necessary in order to get accurate results at different distances? (H. 42, 1885.)

L.—"Reflection of light from plane mirrors. Parallel mirrors. Inclined mirrors. Concave and convex spherical mirrors. Formation of images. Real and virtual images. Conjugate foci. Position and size of images."

1. What must be the angle between two plane mirrors in order that an incident ray which is parallel to one of them may, after two reflections, be parallel to the other? (26, 1886.)

M.—"Refraction of light at plane surfaces. Index of refraction. Refraction through a plate. Multiple images."

"Images formed by refraction."

"Total internal reflection. Critical angle. Refraction through a prism."

"Refraction at spherical surfaces."

"Formation of images by lenses. Conjugate foci."

"Real and virtual images."

"The principles of the telescope, the opera glass."

1. Light (of one colour only) radiates from a point on the axis of a large convex lens, and converges on the other side of the lens. Describe and explain the effect of interposing a "stop" (*i.e.*, an opaque screen with circular aperture) between the source of light and the lens. (27, 1885.)

2. A convex lens of 6 inches focal length is employed to read the graduations of a scale, and is held so as to magnify them three times. Find how far it is held from the scale. (29, 1885.)

3. Explain why a fish seen in a pond or in an aquarium tank appears to be nearer the observer than it really is. If the refractive index of the water be 1·34, and the apparent distance of the fish from the surface be 10 inches, what will be its true distance? (30, 1885.)

4. A ray of light is incident perpendicularly upon one of the two faces of a right-angled isosceles prism which bound the right angle Draw a picture showing the subsequent path of the ray, and give reasons for your construction. (28, 1886.)

5. State generally the effect of a lens upon a ray of light passing through it. Show how with a double convex lens an image of a lighted candle may be seen (1) inverted and magnified, (2) inverted and diminished, (3) erect and magnified. (29, 1886.)

6. Explain how to determine the focal length of a double convex lens without the aid of sun light. (26, 1887.)

7. Explain how to measure the refracting angle of a prism, and the refractive index of the material of the prism. (27, 1887.)

8. A convergent pencil of light falls upon a concave lens. Trace the position of the image as the point of convergence of the pencil moves from an infinite distance up to the lens. (30, 1887.)

9. Describe the measurements which are required, and the method of determining the refractive index, of a transparent substance in the form of a prism. (27, 1888.)

10. Find the magnifying power of a simple lens of one inch focal length, placed close in front of an eye whose distance of distinct vision is 10 inches. Illustrate with a figure. (28, 1888.)

11. Show how to determine, either graphically or arithmetically, the position and magnitude of the image of an object placed in front of a convex lens. An arrow 5 inches long is placed 8 inches away from a convex lens whose focal length is 3 inches. Find the position and length of the image. (25, 1889.)

12. Explain the apparent raising of a picture stuck at the bottom of a cube of glass, till it appears to an eye looking down as if it were in the glass. If the index of refraction is 1·8, how much does the picture appear raised to perpendicular vision? (26, 1889.)

13. Explain how to find the index of refraction of a liquid prism for sodium light, stating clearly what measurements are required. (25, 1890.)

14. Explain the principle of the compound microscope, with the aid of a diagram. From what data, and how, would you calculate its magnifying power? (27, 1890.)

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15. Explain how to determine the focal length, the curvatures, and the index of refraction of a double convex lens by means of the optical images obtained by reflection or by refraction at its surfaces. (H. 41, 1887.)

16. A system of parallel rays is incident on a refracting sphere. Prove that there is a certain angle of incidence for which the deviation is a maximum, when the pencil of rays is once internally reflected. Explain the application of this proposition to the theory of the rainbow. (H. 42, 1888.)

(For *N*, *O*, *P*, see pages 168 and 169.)

Q.—"*The phenomena of the polarization of light. Use of reflectors. Nicol's prisms and tourmaline for the polarization of light. Properties of Iceland spar. Simple forms of polariscope. Interference of polarized light.*"

1. What are the optical characteristics of a rhomb of Iceland spar? Give Huygens' construction for the course of the ordinary and extraordinary rays, and state in what planes these two rays are polarized. What must be the plane of incidence that both rays may be ordinarily reflected? (H. 42, 1886.)

2. Explain the construction of a Nicol's prism.

A ray of light passes through a crystal of Iceland spar, then through a selenite plate, and finally through a Nicol's prism. Explain the changes which it undergoes in its passage through each. (H. 42, 1887.)

3. What are the optical properties of quartz and arragonite?

For what purposes are quartz prisms usually employed? and how should they be cut to avoid inconveniences arising from the optical properties of quartz? (H. 44, 1890.)

4. Describe the saccharometer. Explain the mode of using it to determine the strength of a sugar solution. Mention different forms of analyser which may be used. (H. 43, 1889.)

N.—"The composite character of white light."
"*Achromatic prisms and lenses. The spectroscope. Spectrum analysis.*"
"*Spectra produced by glowing solids, liquids, and gases.*"
"*Absorption of light by substances. Cause of colour.*"
"*Relation between radiation and absorption. Absorption spectra.*"

1. Describe the principal parts of a spectroscope. (30, 1886.)

2. A brightly illuminated vertical slit in the shutter of a dark room is looked at through a prism with a vertical edge. Draw a picture showing how the image seen is formed, and explain why it is coloured. (29, 1887.)

3. An iron ball is made white hot, and its spectrum examined by a spectroscope. What will be the nature of the spectrum seen? Again, a piece of iron is used as one of the poles in the electric arc, and the spectrum examined. In what respects do the two spectra differ from one another? To what molecular conditions do you suppose the difference to be due? (29, 1888.)

4. Sketch the plan of a spectroscope, explaining the use of the collimator; and describe how to make a map of the spectrum of a given substance. (27, 1889.)

5. How may a colourless image of an object be obtained by means of a combination of lenses? Given two substances of refractive indices 1·5 and 1·7, of dispersive powers in the ratio of 3 to 4, what should be the focal lengths of each simple lens in an achromatic combination of 5 feet focus? (26, 1890.)

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6. Explain clearly in what ways the spectroscope has been used to determine the constitution of the sun, and discuss the evidence obtained by its means. (H. 43, 1886.)

O.—"*The wave theory of light. Explanation of reflection and refraction.*"

1. Explain the law of refraction of light at a plane surface according to the wave theory. (26, 1888.)

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2. Give an explanation, in accordance with the wave theory, of the formation of primary and secondary rainbows and of supernumerary bows. (H. 43, 1890.)

P.—" *Interference of light. Newton's rings. Colours of thin plates. Diffraction. The diffraction spectrum.*"

1. A convex lens is pressed on a piece of plate glass. Explain and state the laws of the phenomena which are observed near the point of contact of the lens and plate. (30, 1888.)

2. A beam of white light proceeding from a narrow slit is thrown by means of a lens on to a screen. Explain the appearance which would be produced by the placing of a diffraction grating between the lens and the screen in the path of the beam. (28, 1889.)

3. Describe and explain the appearance when a convex lens is pressed into contact with a piece of plate glass, and a beam of light reflected from their surfaces near the point of contact is focussed on a screen. (28, 1890.)

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4. Explain the principles of some method of determining the lengths of luminous waves by the observation of phenomena due to interference. Describe fully the details of the method. (H. 43, 1885.)

5. Fine parallel lines are ruled closely at regular intervals on a glass plate. Explain precisely why and where spectra are formed when the plate is illuminated by a beam of light coming through a lens from a slit arranged parallel to the ruled lines. Explain the effect on these spectra of the closeness of the ruling and the breadth of the ruled space. (H. 43, 1888.)

6. Describe Newton's rings as seen by reflection. What relations hold between the thickness of the air space, the diameters of the rings, and the angle of incidence of the light? (H. 42, 1889.)

7. Give an account of experiments on the interference of light due to the reflection of light from the front and back faces of thick plates of glass, and describe a method of determining the relative refractive indices of different liquids by such interference. (H. 42, 1890.)

(For *Q*, see page 167.)

ANSWERS.

L. 1. 60°. See page 39.
M. 2. 4 inches.
M. 3. 7·46 inches. See page 56.
M. 10. 11 times.
M. 11. 4·8 on other side of lens and 3 inches long.
M. 12. 4/9ths of its real distance.
P. 6. Air space thicknesses are as natural numbers; diameters of rings are as square roots of the natural numbers. See page 137, where *t* may be taken as the air space, and *d* as radius of the corresponding ring.

SUB-SUBJECT III.—HEAT.

S.—" Distinction between heat and temperature."

" Construction of mercurial thermometer. Different kinds of thermometers. Different thermometric scales."

" Expansion of solids. Linear and cubical expansion. Effects of unequal expansion. Compensated pendulums."

" Expansion of liquids. Absolute and apparent expansion "

" Peculiarity in the expansion of water."

" Expansion of gases. Comparison of the volumes of gases at different temperatures. The air thermometer."

" *Methods of measuring the expansion of solids, liquids, and gases.*"

1. Describe a method of finding the coefficient of increase of elasticity of a mass of air occupying a constant volume. (31, 1885.)

2. A glass bottle holds when quite full at the temperature of melting ice 20 cubic inches of ice-cold water. How many cubic inches of boiling water will it hold, the bottle as well as the water being at 100°C. ?

(Coefficient of linear expansion of glass = 0·000009.) (32, 1885.)

3. A hundred cubic inches of air weigh 31 grains when the temperature is 0°C. What will be the weight of the same volume of air when the temperature is 20°C and the pressure one-third of its original amount, the coefficient of expansion being 11/3000 ? (32, 1886.)

4. Describe the construction and action of maximum and minimum thermometers. (34, 1886.)

5. Distinguish between the apparent and the absolute expansion of mercury. Describe some method by which one *or* the other may be experimentally determined. (33, 1887.)

6. How can the absolute expansion (*a*) of mercury, (*b*) of any other liquid, be determined ? (34, 1888.)

7. State concisely the characteristic properties of permanent gases, and sketch an apparatus for determining their coefficient of expansion at constant volume. (32, 1889.)

8. Distinguish between the coefficient of apparent expansion and the coefficient of absolute expansion of a liquid. By what method has the coefficient of absolute expansion of mercury been determined ? (29, 1890.)

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9. Describe the process of determining the coefficient of expansion of a liquid like alcohol or paraffin, remembering the necessary preliminary determination of the expansibility of the glass vessel employed. (H. 45, 1889.)

T.—"Change of state from solid to liquid, and from liquid to gas."

"Fusion and solidification Latent heats. Freezing mixtures. Alteration of the volume of water and other substances on solidification. Effects of frost."

"Boiling. Latent heat of vaporization. Effect of pressure on the boiling point."

"Evaporation. Vapour pressure. Saturation. Maximum pressure of a vapour. Cold produced by evaporation. Mixture of gases and vapours."

"Condensation of aqueous vapour. The dew-point. Means of determining it. The hygrometric state of the air. The wet and dry bulb hygrometer. Condensation hygrometers."

"Clouds, rain, snow, and hail."

1. Describe the changes of volume and temperature which a pound of melting ice undergoes when heat is continually applied to it until it becomes steam. (33, 1885.)

2. Explain what is meant by the dew-point. How does the dew-point show the amount of vapour in the atmosphere? Describe the mode of action of a dew-point hygrometer. (31, 1886.)

3. What is meant by "latent heat of vaporization?" If the latent heat of vaporization of water be 966 when one degree Fahrenheit is the unit of temperature, what will it be when one degree centigrade is the unit? Would your result be different if the unit of mass had been changed? (See U 3 and Y 1.) (33, 1886.)

4. Describe the method of using a dew-point hygrometer to determine the hygrometric state of the air in a room. (34, 1887.)

5. State the laws of fusion. What is the effect of pressure in changing the temperature at which water freezes? (35, 1887.)

6. Define what is meant by "the latent heat" of water; and state exactly how you will proceed to measure it experimentally. (31, 1888.)

7. What is the meaning of the term "humidity," or "hygrometric state," as applied to the atmosphere? Give a short description of the instruments that are commonly employed for finding the state of the air with respect to moisture. The dew-point on a certain day being found to be 12°C., and the temperature of the air being 16·5°C., find its humidity, given that the maximum pressure of aqueous vapour at 12° is equivalent to 1·046 cm. of mercury, while that at 16° is 1·364 cm., and that at 17° is 1·442 cm. (35, 1888.)

8. Describe an experiment showing that water can be frozen by its own evaporation What weight of vapour must evaporate in order to freeze a gramme of water already at freezing point? (30, 1890.)

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9. A quantity of dry air measures 4500 c.cm. at 10°C. and 760 mm. pressure. If the air be heated to 33°C., and saturated with vapour at that temperature, find the volume of the mass of air in order that the pressure may be doubled, the elastic force of vapour at 33°C. being represented by a column of 37·41 mm. of mercury. (H. 44, 1885.)

10. Describe Andrews' researches on carbonic acid at various temperatures and pressures. Draw the isothermal lines for this substance, and explain the terms *critical temperature* and *critical pressure.* (H. 43, 1887.)

U.—"Specific heat calorimeters. Methods of determining specific heats. Consequences of the high specific heat of water."

1. What is meant by the specific heat of a substance? A copper calorimeter of specific heat 0·095 has a mass of 120 grammes, and contains 280 grammes of water at 15°C. Find the specific heat of a substance when 375 grammes of it, at a temperature of 100°C., will when immersed raise the temperature of the water to 25°C. (34, 1885.)

2. Define specific heat.

A copper calorimeter having a mass of 125 grammes contains 400 grammes of water at 15°C.; into this are placed 556 grammes of copper at a temperature of 95°C., and the final temperature is 24°C. Find the specific heat of copper. (31, 1887.)

3. Describe a mode of determining the specific heat of a solid. A mass of 700 grammes of copper at 98°C., put into 800 grammes of water at 15°C., contained in a copper vessel weighing 200 grammes, raises the temperature of the water to 21°C. Find the specific heat of copper. (29, 1889.)

4. Describe any good method of determining the specific heat of a solid substance.

The latent heat of tin is 14·25, its specific heat ·064, and its melting point 232° in the centigrade scale. What will be the corresponding numbers in the Fahrenheit scale? (31, 1890.)

V.—"Convection of heat in liquids and gases: Heating buildings by hot water. The Gulf Stream and ocean currents."

W.—"Conduction of heat. Effects due to the high or low conducting power of various substances."

1. Describe the experiments necessary for determining the thermal conductivity of a metal like copper, and point out clearly how the value of the thermal conductivity is deduced from the data of the experiment. (H. 45, 1886.)

X.—"The reflection, radiation, absorption, and transmission of heat by various substances."

"*Laws of cooling.*"

1. Give an account of Dulong and Petit's investigations on the laws of cooling of a hot body in vacuo. (H. 44, 1887.)

Y.—"*Relation between heat and energy. The mechanical equivalent of heat.*"

"*The specific heats of gases at constant volume and at constant pressure. The bearing of these specific heats on the velocity of sound, and on the mechanical equivalent of heat.*"

"*Laws of thermodynamics.*"

"*Changes in the melting points of bodies produced by pressure.*"

1. What is meant by *energy?* If the mechanical equivalent of heat be 1390 *foot-pounds-centigrade*, through what height ought a quantity of water weighing 2 lb. to fall so that its mechanical energy when it reaches the ground may be equivalent to the heat required to raise it in temperature *one degree Fahrenheit?* (35, 1885.)

2. Explain the difference between specific heat of air at constant pressure and its specific heat at constant volume. Which of them is the greater, and why? (35, 1886.)

3. What is meant by saying that the mechanical equivalent of heat is 772 foot-pounds? If the standard substance were iron (whose specific heat is 0·114), what would be the value of the mechanical equivalent of heat? (32, 1887.)

4. What is meant by saying that heat is a form of energy? How has the amount of energy corresponding to a unit of heat been determined? (32, 1888.)

5. The temperatures in the boiler and condenser of a steam engine are 150° and 50° respectively. What proportion of the heat supplied to the engine will be utilized if the engine is a perfect engine? (33, 1888.)

6. Explain in what way the melting points of bodies are affected by pressure. Illustrate your answer by reference to the case of water and wax. (30, 1889.)

7. What do you understand exactly by the efficiency of a steam engine? What is the theoretical efficiency of a steam engine whose boiler is at 150°C. and its condenser at 40°? (31, 1889.)

8. Explain the adhesion of two pieces of melting ice when pressed together and let go. Why is it not readily possible to make snowballs during frost? (32, 1890.)

9. Discuss the various methods of determining the mechanical equivalent of heat.

Determine the value of the mechanical equivalent from the following data :—Specific heat of air at constant pressure = 0·2374. Velocity of sound in air at 0°C. = 33,300 (cm. sec.). Mass of 1 c.cm. of dry air at 0°C. and 760 mm. pressure = 0·001293 gramme.

Coefficient of expansion of air = $\frac{1}{273}$, $g = 981$.

Density of mercury at 0°C. = 13·595. (H. 45, 1885.)

10. Distinguish between the isothermal and the adiabatic elasticity of a gas, and show their relation to each other. Hence explain how the velocity of sound in the gas may be calculated. In the case of atmospheric air, point out what data are necessary for the calculation. (H. 45, 1887.)

11. Explain the difference between the adiabatic and the isothermal elasticities of a substance; and show how to determine the ratio of these two elasticities for common air experimentally. (H. 44, 1888.)

12. Explain exactly the adhesion of two pieces of ice when pressed together at the melting point and then released. Calculate the effect of an extra atmosphere of pressure on the melting point of a substance which contracts on solidifying by one-sixth of its volume in the liquid state, whose latent heat is 40 units, whose ordinary freezing point is 27°C., and whose specific gravity when liquid at this temperature is 1·2. (H. 45, 1888.)

13. If a quantity of air at 15°C. is suddenly compressed to half its volume, show how to calculate the temperature it will momentarily attain. Explain the importance of this knowledge in the theory of sound-propagation.

Show how a numerical answer can be given with the help of the following data :—

log 2 = 0·3010
log 1·32 = 0·1204. (H. 41, 1889.)

14. Explain how the thermal equivalent of mechanical energy may be derived from a knowledge of how much heat is required to warm a given quantity of gas a given number of degrees, first when the pressure on it is kept constant, and next when it is heated in a closed inexpansible vessel. (H. 44, 1889.)

15. Show how to find the specific heat of air at constant volume, when its specific heat at constant pressure and the mechanical equivalent of heat are known.

When a mass of air is heated (1) at constant pressure, (2) at constant volume through the same range of temperature, how would you represent on an indicator diagram the mechanical equivalents of the quantities of heat required for the two operations? (H. 45, 1890.)

16. What is meant by the critical temperature, pressure, and volume of a gas? How can these be determined experimentally?

Draw the isothermals of a substance at temperatures above and below the critical temperature. (H. 46, 1890.)

Z.—Not classed.

1. Describe the methods which have been employed for determining the amount of heat developed by the union of oxygen and hydrogen in forming water. (H. 44, 1886.)

ANSWERS.

S. 2. 20·054 c. in. **S. 3.** 9·627 grains nearly.
T. 3. 536·6. No. **T. 7.** 73·8. **T. 9.** 2494 c.cm. nearly.
U. 1. 0·1036. **U. 2.** 0·0938. **U. 3.** 0·09108.
Y. 3. 88 nearly. **Y. 5.** 23·64 per cent. nearly. **Y. 7.** 26 per cent. nearly.
Y. 11. $4·317 \times 10^7$. **Y. 12.** 0·248. Temperature raised. **Y. 13.** 103°·16C.

SERIES II.

UNIVERSITY OF LONDON

INTERMEDIATE, HONOURS, & DEGREE EXAMINATIONS.

NOTE.—The Questions are selected from the following Examination Papers:—

INTERMEDIATE SCIENCE (PASS).	INT.SC.
BACHELOR OF SCIENCE (PASS).	B.SC.

The Questions extend over the period 1887-1889, inclusive.

SOUND.

1. A string stretched with a weight of 25 lb. when made to vibrate transversely gives a certain note. What tension must be applied to a string of the same material, but of twice the length and twice the thickness, to make it give the octave above that note? (Int.Sc., 1887.)

2. What are the physical properties in a vibrating body which correspond to the intensity, pitch, and quality of the sound produced? (Int.Sc., 1888.)

3. How would you verify the relation between the pitch of the note emitted by a stretched string and the tension? Four exactly similar and equal strings stretched with the same tension are vibrating side by side. How will the note emitted be affected if they be fastened together so as to form one string by winding round them an extremely thin piece of silk? (Int.Sc., 1888.)

4. State and explain carefully how you suppose the air in an open organ pipe which is sounding its fundamental note to be moving, defining the terms wave length, frequency, amplitude. What is the relation between the length of the pipe and the wave length of the note sounded? (Int.Sc., 1889.)

5. Describe some method of measuring the velocity of sound in a rod of glass. (Int.Sc., 1889.)

6. What are sensitive flames? and what is the cause of their sensitiveness? What relation exists between the pitch of the note emitted by a flame in a cylindrical tube and the length of the tube? and what experiments would you make to establish the relation? (B.Sc., 1887.)

7. Show how to find the velocity of sound in a brass rod from observations on the pitch of its longitudinal vibrations. (B.Sc., 1888.)

8. How would you show that the quality of the note emitted by a stretched string depends on the number and relative intensities of the overtones present? (B.Sc., 1888.)

9. Describe some form of apparatus for showing the combination of two rectilinear simple harmonic vibrations taking place at right angles to each other. If one vibration be the octave of the other, trace the form of the resulting curve. (B.Sc., 1889.)

10. Define a musical interval, and explain clearly why keyed instruments are never tuned to the correct intervals of theoretical scale. (B.Sc., 1889.)

LIGHT.

1. Describe Foucault's method of measuring the velocity of light. If red and blue light travelled with different velocities in air, how would the appearance presented to the observer be modified? (Int.Sc., 1887.)

2. A small air bubble in a sphere of glass 4 inches in diameter appears when looked at so that the bubble and the centre of the sphere are in a line with the eye to be 1 inch from the surface. What is its true distance? $\mu = 1{\cdot}5$. (Int.Sc., 1887.)

3. Explain why a thick plate of glass produces an appreciable displacement in the apparent position of a near object reviewed through the plate, but an inappreciable displacement for distant objects. (Int.Sc., 1888.)

4. Define the focal length of a spherical reflecting surface. How far from a concave mirror of radius 3 feet would you place an object to give an image magnified three times? Would the image be real or virtual? (Int.Sc, 1888.)

5. What do you understand by the intensity of illumination at a point? and how would you show that the intensity of illumination at a point, due to a given source, is inversely proportional to the square of the distance of the point from the source? (Int.Sc., 1889.)

6. Explain the action of a lens when used as an eyeglass. A man who can see most distinctly at 5 inches from his eye, wishes to read at a distance of 15 feet off. What sort of spectacles must he use? and what must be their focal length? (Int.Sc., 1889.)

7. Explain fully without calculation how a combination of two prisms can be constructed which shall be achromatic for two given colours. Show that such a combination will not be achromatic for all colours. (B.Sc., 1887.)

8. Describe the astronomical telescope fitted with Ramsden's eye-piece, and draw a figure showing the path of a pencil of rays from a distant object through it. What advantage has Ramsden's over Huyghen's eye-piece? and why is the latter usually employed in microscopes? (B.Sc., 1887.)

9. Describe the appearance presented to a person looking at a small object through two rhombs of Iceland spar, and the changes which take place as one rhomb is turned round the line of vision. How are they theoretically explained? (B.Sc., 1887.)

10. Describe the experiment in which interference of light is produced by means of Freshnel's bi-prism. How do the phenomena observed vary with the wave length of the light employed? and how is the wave length in any case determined? (B.Sc., 1887.)

11. Describe the eye as an optical instrument, and show how the defects of long and short sight may be corrected. An observer looks at a very bright object and then at a sensitive photographic plate in a dark room. On developing the plate an image of this bright object is found, due to phosphorescence of the retina. Explain with figures the course of the rays by which this image is produced. (B.Sc., 1888.)

12. Describe and explain the arrangement of the apparatus required for the production of the pure spectrum of an electric spark. How would you arrange the apparatus to examine in detail the light from each portion of the spark, *e.g.*, to compare the spectra from the parts near the two poles respectively? (B.Sc., 1888.)

13. How does the wave theory account for the existence of rays of light, and for their reflection and refraction? (B.Sc., 1888.)

14. A conical beam of plane polarized light falls nearly normally upon a plate of Iceland spar cut at right angles to the axis. This is viewed through a Nicol's prism. Describe and explain the appearances presented as the Nicol is rotated on its axis. (B.Sc., 1888.)

15. What is meant by the chromatic aberration of a lens? and how is it corrected in the object glass of a telescope? The mean refractive indices of two specimens of glass are 1·52 and 1·66 respectively; the differences in the indices for the same two lines of the spectrum are 0·013 for the first, and 0·022 for the second. Find the focal length of a lens of the second glass which, when combined with a convex lens of 50 cm. focal length of the first, will make an object glass achromatic for these two lines. (B.Sc., 1889.)

16. Describe fully experiments to show under what circumstances an incandescent body gives (*a*) a continuous spectrum, (*b*) a spectrum of bright lines, (*c*) a luminous spectrum with dark lines. (B.Sc., 1889.)

17. Describe experiments to show that two lights may interfere and produce darkness. Why are the lights from two candles never seen to interfere? (B.Sc., 1889.)

18. Explain the action of Nicol's prism. You are given two such prisms; show how to place the second so that the light which emerges from it may be half as intense as that which falls on it. (B.Sc., 1889.)

HEAT.

1. What is an air thermometer? How is it constructed? and how is it used? What means have we besides the air thermometer of measuring temperatures between 400° and 800°C.? (Int.Sc., 1887.)

2. Given two similar bars of gold and silver, describe an experiment by which their relative thermal conductivities may be determined. (Int.Sc, 1887.)

3. One hundred cubic centimetres of oxygen saturated with water are collected at a pressure of 740 mm. and a temperature of 15°C. Find the volume of dry oxygen at 0° and 760 mm., having given that the maximum pressure of aqueous vapour at 15° is 12·7 mm. *Ans* 78·24 c.cm. (Int.Sc., 1887.)

4. Explain carefully the statement that the latent heat of fusion of water is 80. What is the unit in terms of which latent heat is measured? Trace the changes in the temperature and volume of a kilogramme of ice at −5°C. to which heat is applied until it is converted into steam. (Int.Sc., 1888.)

5. How are the radiating and absorbing powers of a surface connected? Describe experiments to verify this connection. (Int.Sc., 1888.)

6. A glass bulb with a uniform fine stem weighs 10 grammes when empty, 117·3 grammes when the bulb only is full of mercury, and 119·7 grammes when a length of 10·4 cm. of the stem is also filled with mercury. Calculate the relative coefficient of expansion for temperature of a liquid which, when placed in the same bulb, expands through the length from 10·4 to 12·9 cm. of the stem when warmed from 0°C. to 28°C. The density of mercury is 13·6 grammes per c.cm. (Int.Sc., 1889.)

7. A mass of 200 grammes of copper whose specific heat is 0·095 is heated to 100°C., and placed in 100 grammes of alcohol at 8°C. contained in a copper calorimeter whose mass is 25 grammes, and the temperature rises to 28°·5C. Find the specific heat of the alcohol. (Int Sc., 1889.)

8. Distinguish between a gas and a vapour. Describe the behaviour of a mass of carbonic acid during the following operations:—(1) the temperature being 15°C., the pressure is gradually increased from 1 to 50 atmospheres; (2) at this pressure the temperature is gradually raised to 40°C.; (3) the pressure is then slowly reduced to atmospheric. (B.Sc., 1887.)

9. The heat of combustion of zinc in oxygen is said to be 1300. Explain what this statement means, and describe the apparatus and experiments by which it may be verified. (B.Sc., 1887.)

10. Describe and explain the action of Crooke's radiometer. (B.Sc., 1887.)

11. What is meant by the dissipation of energy? and how is the doctrine of dissipation of energy connected with the second law of thermodynamics? (B.Sc., 1887.)

12. Equal known quantities of oil and water are placed in two calorimeters, and a coil of wire of unknown resistance is inserted in each. Currents of electricity are passed through the wires, the currents being adjusted so that the rise of temperature is the same in each of the two calorimeters. Show how to determine hence the specific heat of the oil, and explain what further data are requisite to make the measurement exact. (B.Sc., 1887.)

13. Describe carefully an accurate method of measuring the coefficient of expansion by heat of a solid, including the corrections and precautions that would probably be required in practice. (B.Sc., 1888.)

14. Describe carefully how to measure accurately the coefficient of absorption of a transparent solid for radiant energy. (B.Sc., 1888.)

15. What experiments are necessary to determine the relation between the thermal conductivity of a body and its temperature? How may the thermal conductivity of a gas be measured? (B.Sc., 1888.)

16. Explain clearly how it follows from the theoretical efficiency of a reversible engine that ice is melted by pressure. (B.Sc., 1888.)

17. Explain how the specific heat of air at a constant pressure has been determined. (B.Sc., 1888.)

18. State Dalton's law as to the pressure of a mixture of gases or vapours. By what experiments would you verify it for air and water vapour at a temperature of about 50°? (B.Sc., 1889.)

19. Describe experiments by which the laws of cooling may be found, and give an account of the results obtained. (B.Sc., 1889.)

20. Explain how to measure the conductivity for heat of a badly-conducting substance, such as rock or sand. (B.Sc., 1889.)

21. What do you understand by a reversible heat engine? and what is meant by the efficiency of a machine? How would you show that the efficiency of all reversible heat engines, working between the same limits of temperature, is the same? (B.Sc., 1889.)

22. Show how to calculate the mechanical equivalent of heat from a knowledge of the specific heats of air at constant pressure and volume respectively, explaining carefully any assumption made in the process. (B.Sc., 1889.)

ANSWERS *(Numerical only)*.

SOUND. **1.** 1600 lb. **2.** It will be an octave lower.

LIGHT. **2.** 1·2 inch. See page 63. **4.** 2 inches when image is real, 1 inch when virtual. **6.** Concave. $f = 5\frac{1}{7}$ inch. **15.** 66·6 cm.

HEAT. **3.** 78·24 c.cm. **6.** 0·00879. **7.** 0·639 nearly.

TABLES AND CONSTANTS.

Velocity of Sound in Gases at 0°C.

	Feet per Second.	Metres per Second.		Feet per Second.	Metres per Second.
Air	1092*		Carbonic acid	858	261
Oxygen	1040	317	Carbonic oxide	1107	337
			Nitrous oxide	859	262
Hydrogen	4164	1269	Olefiant gas	1030	314

Column 1. From Tyndall's *Sound*, 1883, page 36.

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* 1090 feet per second at 0°C. is given by Sir John Herschell, ***Encyclopædia Metropolitana***, as the probable velocity of sound in air at 0°C., and this number is commonly adopted.

Velocity of Sound in Liquids.

(M. Wertheim, given in Tyndall's *Sound*, page 37.)

Name of Liquid.	Temperature.	Velocity.
		Feet.
River water (Seine)	15°C.	4714
,, ,,	30	5013
,, ,,	60	5657
Sea water (artificial)	20	4768
Solution of common salt	18	5132
,, chloride calcium	23	6493
Absolute alcohol	23	3804
Common ,,	20	4218
Spirits of turpentine	24	3976
Sulphuric ether	0	3801

Note from above table, (*a*) that sound travels faster in warm than in cold water; (*b*) that salts augment the velocity.

Velocity of Sound in Metals.

(Wertheim, quoted by Tyndall.)

Name of Metal.	At 20°C.	At 100°C.	At 200°C.
Lead	4030	3951	
Gold	5717	5640	5619
Silver	8553	8658	8127
Copper	11666	10802	9690
Platinum	8815	8437	8079
Iron wire	16130	16728	
Cast steel	16357	16153	15709
Steel wire (English)	15470	17201	16394

Presumably the numbers refer to feet per second.

Refractive Indices (approximate*).

Diamond	2·479	Fluor spar	1·436
Garnet	1·815	Calcite (ord.)	1·658
Sapphire	1·768	,, (ex.)	1·487
Topaz	1·610	Canada balsam	1·530
Flint glass	1·642	Ice	1·310
Crown glass	1·533	Water	1·333
Quartz	1·548	Carbon disulphide	1·631
Rock salt	1·498	Air	1·0003
Alum	1·457		

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* For detailed indices of refraction for various rays the student is referred to "Units and Physical Constants," by Everett; or to *Physikalisch-chemische Tabellen*, by Laudolt and Börnstein.

Refractive Indices of Gases. (Everett.)

(Sodium light, at 0°C. and 760 mm.)

Air	1·0002927	Nitrous oxide	1·0005159
Nitrogen	2977	Carbonic oxide	3350
Oxygen	2706	Carbon dioxide	4544
Hydrogen	1387	Cyanogen	8216

Index of Refraction for differently coloured Rays (Fraunhofer lines) of the Spectrum. (Fliedner.)

Medium.	U. Red. B λ 6867	Red. C λ 6562	Yellow. D λ 5895 5889	Green. E λ 5269	B. Green. F λ 4861	Violet. G λ 4307	Violet. H λ 3967
Flint glass	1·628	1·630	1·635	1·642	1·648	1·660	1·671
Crown ,,	1·526	1·527	1·530	1·533	1·536	1·542	1·547
Water	1·331	1·332	1·334	1·336	1·338	1·341	1·344

λ = wave length of ray in ten-millionths of a millimetre.

Linear Coefficients of Expansion. (Fliedner.)

Substance.	Expansion. 0° to 100°C.	
Lead	0·002848	1/351
Wrought iron	0·001167	1/856
Cast Iron.	0·001110	1/901
Glass	0 000861	1/1161
Gold	0·001401	1/645
Wood	0·000380	1/2631
Soft solder	0·002505	1/399
Copper	0·001717	1/582
Marble	0·000849	1/1178
Brass	0·001920	1/521
Platinum.	0·000856	1/1167
Sandstone	0·001174	1/852
Silver	0·001909	1/524
Steel (hard)	0·001225	1/816
,, (soft)	0·001079	1/926
Zinc (rolled)	0·003331	1/302
,, (cast)	0·002987	1/336
Zinc solder (1 zinc, 2 copper)	0·002058	1/436
Tin.	0·002173	1/450
Brick or tile	0·000500	1/2000

Between – 27°·5C. and – 1°·25C. ice expands 0·000052 of its length for each degree.

Cubic Expansion (Absolute) of Mercury. (Regnault.)

A volume of 1·000000 at 0°C. becomes	at 50° 1·009013	at 100° 1·018153	at 150° 1·027419
	at 200° 1·036811	at 250° 1·046329	at 300° 1·055973

Average coefficient for 1° between 0° and 100° = 1·00018153.

Volume and Density of Water at different temperatures.

(Given as mean results of many observations.)

Watt's *Dictionary of Chemistry*, Vol. VI., page 672.

Temp.	Sp. gr. of Water (at 0°C. = 1).	Vol. of Water (at 0°C. = 1).	Sp. gr. of Water (at 4°C. = 1).	Vol. of Water (at 4°C. = 1).
0°C.	1·000000	1·000000	0·999871	1·000129
1	1·000057	0·999943	·999928	1·000072
2	1·000097	·999902	·999969	1·000031
3	1·000120	·999880	·999991	1·000009
4	1·000129	·999871	1·000000	1·000000
5	1·000119	·999881	0·999990	1·000010
6	1·000099	·999901	·999970	1·000030
7	1·000062	·999938	·999933	1·000067
8	1·000015	·999985	·999886	1·000114
9	0·999953	1·000047	·999824	1·000176
10	·999876	1·000124	·999747	1·000253
11	·999784	1·000216	·999655	1·000345
12	·999678	1·000322	·999549	1·000451
13	·999559	1·000441	·999430	1·000570
14	·999429	1·000572	·999299	1·000701
15	·999289	1·000712	·999160	1·000841
16	·999131	1·000870	·999002	1·000999
17	·998970	1·001031	·998841	1·001160
18	·998782	1·001219	·998654	1·001348
19	·998588	1·001413	·998460	1·001542
20	·998388	1·001615	·998259	1·001744

Specific Heats of various Solids. (Kopp and others.)

Aluminium	0·202	Brass	0·0939
Antimony	0·0508	Glass	0·1977
Bismuth	0·0305	Ice	0·5050
Cadmium	0·0542	Iodine	0·0541
Copper	0·0930	Diamond (−50°·5C.)	0·0635
Gold	0·0324	,, (+10°·7C.)	0·1128
Iron	0·112	,, (+985°C.)	0·4589
Lead	0·0315	Bromine (solid)	0·0843
Magnesium	0·245	Beryllium or Glucinum (Humpidge) 100°	0·4702
Mercury (solid)	0·0319	200°	0·540
Platinum	0·0325	400°	0·6172
Silver	0·0560	500°	0·6206
Tin	0·0548		
Zinc	0·0932		

Specific Heat of Water.

	Mean Sp. Ht.	Sp. Ht.		
		at	0°	1·0000
Between 0° and 10°C.	1·0002	,,	10	1·0005
,, 0 ,, 20	1·0005	,,	20	1·0012
,, 0 ,, 40	1·0013	,,	40	1·0030
,, 0 ,, 60	1·0023	,,	60	1·0056
,, 0 ,, 80	1·0035	,,	80	1·0089
,, 0 ,, 100	1·0050	,,	100	1·0139
,, 0 ,, 200	1·0160	,,	200	1·0440

Fusing Constants and Fusing Temperatures.

(W. H. Preece. *Nature*, May 22nd, 1890, page 94.)

For explanation see page 189.

	Fusing Constant (*a*).	Fusing Temperature.		Fusing Constant (*a*).	Fusing Temperature.
Copper	2530	1054	Platinoid	1173	1300
Silver	1900	954	Iron	774·4	1600
Aluminium	1873	650	Tin	405·5	226
German Silver	1292	1200	Lead	340·6	335
Platinum	1277	1775	Alloy (lead 2, tin 1)	325·5	180

Melting Points and Heats of Fusion. (Fliedner.)

Substance.	Melting Point. °C.	Latent Heat.	Substance.	Melting Point. °C.	Latent Heat.
Mercury	− 39	2·83	Zinc	450	28·13
Bromine	− 7·5		Antimony	440	
Ice	0	79·25	Magnesium (Meyer)	800	
Butter	+ 32		Silver	1000	21·07
Phosphorus	44·2	5·034	Copper	1050	
White wax	68	43·54	Gold	1200	
Iodine	107		Cast iron	1100 − 1200	
Sulphur	111	9·368	Steel	1300 − 1400	
Tin	235	14·252	Wrought iron	1600	
Lead	335	5·369	Platinum	1700	
Bismuth	265	12 64			

Rose's metal (1 lead, 1 tin, 2 bismuth) melts at 94°C. Wood's metal (2 lead, 1 tin, 4 bismuth, 1 cadmium) melts at 65°·5C.

Boiling Points and Latent Heats. (Fliedner in part)

Substance.	Boiling Point. °C.	Latent Heat of Vapour.	Substance.	Boiling Point. °C.	Latent Heat of Vapour.
*Oxygen	− 120		Wood spirit	59	263·7
Nitrous oxide	− 88		†Water	100	536·17
Carbonic acid	− 78		Turpentine	293	74·04
Sulphurous anhydride	− 10	95	Phosphorus	290	
Ether	+ 35	90	Linseed oil	316	
Bromine	+ 63	97	Sulphuric acid	325	100
Alcohol	78·4	208·3	Mercury	350	62

---

* At about 450 atmospheres pressure.—See *Memorie sur la Liquefaction de l'oxygene*, by Pictet, pages 69, 70.

† Regnault gives 636·67 as the heat to convert 1 gramme water at 0°C. to steam at 100° mean of numerous experiments. *Chemical Reports and Memoirs*, Cavendish Society, page 273. If from this we subtract 100·5, the heat to raise a gramme of water to 100° (Regnault), we have 536·17 as the latent heat of steam at 100°. Regnault's formula, $\lambda = 606{\cdot}5 + {\cdot}305t$, gives the total heat λ required to convert 1 gramme of water from 0°C. into 1 gramme of saturated steam at t°.

537 is commonly taken as the latent heat of steam.

Tension of aqueous vapour in millimetres.

Temp. C.	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
0·	4·600	633	667	700	733	767	801	836	871	905
1·	4·940	975	*011	*047	*082	*118	*155	*191	*228	*265
2·	5·302	340	378	416	454	491	530	569	608	647
3·	5·687	727	767	807	848	889	930	972	*014	*055
4·	6·097	140	183	226	270	313	357	407	445	490
5·	6·534	580	625	671	717	763	810	857	904	951
6·	6·998	*047	*095	*144	*193	*242	*292	*342	*392	*442
7·	7·492	544	595	647	699	751	804	857	910	964
8·	8·017	072	126	181	236	291	347	404	461	517
9·	8·574	632	690	748	807	865	925	985	*045	*105
10·	9·165	227	288	350	412	474	537	601	665	728
11·	9·792	857	923	989	*054	*120	*187	*255	*322	*389
12·	10·457	526	596	665	734	804	875	947	*019	*090
13·	11·162	235	309	383	456	530	605	681	757	832
14·	11·908	986	*064	*142	*220	*298	*378	*458	*538	*619
15·	12·699	781	864	947	*029	*112	*197	*281	*366	*451
16·	13·536	623	710	797	885	972	*062	*151	*241	*331
17·	14·421	513	605	697	790	882	977	*072	*167	*262
18·	15·357	454	552	650	747	845	945	*045	*145	*246
19·	16·346	449	552	655	758	861	967	*073	*179	*285
20·	17·391	500	608	717	826	935	*047	*159	*271	*383
21·	18·495	610	724	839	954	*069	*187	*305	*423	*541
22·	19·659	780	901	*022	*143	*265	*389	*514	*639	*763
23·	20·888	*016	*144	*272	*400	*528	*659	*790	*921	†
24·	22·184	319	453	588	723	858	996	*135	*273	*411

Temp. C°.	Mm.	°	Mm.	°	Mm.	°	Mm.	°	Mm.
25·0	23·550	25·5	24·261	26·0	24·988	26·5	25·738	27·0	26·505
·1	·692	·6	·406	·1	25·138	·6	·891	·1	·663
·2	·834	·7	·552	·2	·288	·7	26·045	·2	·820
·3	·976	·8	·697	·3	·438	·8	·198	·3	·978
·4	24·119	·9	·842	·4	·588	·9	·351	·4	27·136

The first part of the table is arranged in the same way as logarithm tables, thus the tension at 21·4° C. is 18·954 mms., at 21·5 it is not 18·069 mms., but 19·069 mms.

† Tension at 23·9° C. = 22·058 mms.

Boiling Points and Tensions of Vapour.

Temp. °C.	Alcohol boils 78° mm. Hg.	Ether boils 35° mm. Hg.	Carbon disulph. boils 47° mm. Hg.	Chloroform boils 63°·5 mm. Hg.	Mercury Boils 357° mm. Hg.
0	12·8	183·3	132		0·02
10	24·3	286·4	203		·03
20	44·5	433·3	301·8	160·5	·04
40	133·6	909·6	617	366·2	·08
60	350·3	1728·5	1163·7	751·0	·16
80	812·8	3024·4	2033·8	1404·6	·35
100	1694·9	4950·8	3329·5	2426·5	·75

Tension of mercury vapour at 200° = 19·9 mm., at 300° = 242·1, at 350° = 663·2, at 400° = 1587°·9, at 500° = 6520·2.

Thermal Conductivity of Bodies (Thomson, *Enc. Brit.*); or, calories per second, which would be transferred from one side to the other through one square centimetre section of a plate of the substance one centimetre thick, there being a difference of one degree in temperature between the two sides of the plate.

Copper . .	0·96	Sand. . .	0·0026
Iron. . .	0·2	Water . .	0·002
Air . . .	0·000049	Fir across fibres.	0·00026
Oxygen . .		Fir along fibres .	0·00047
Nitrogen . .		Cork. . .	0·000029
Carbonic oxide .		Carded wool .	0·000122
Hydrogen. .	0·00034	Cotton wool .	0·000111
Sandstone. .	0·01	Eider down .	0·000108

Specific Heat of Gases.*

Name of Gas or Vapour.	Density Air = 1.	Sp. Ht. of Equal Weights (Water = 1) or calories required to raise one gramme 1°C.		Sp. Ht. of equal volumes, expressed in calories, required to raise 773 c.cm. 1°C.	$\frac{c_p}{c_v}$
		Under constant pressure. c_p	At constant volume. c_v		
Air	1·0000	0·2374	0·168	0·2374	1·413
Oxygen	1·1056	·2175	·155	·2405	1·403
Nitrogen	0·9713	·2438	·173	·2370	1·409
Hydrogen	0·0692	3·4090	2·406	·2359	1·417
Chlorine	2·4400	·1210	·093	·2962	
Nitrous oxide	1·5250	·2238	·180	·3447	1·243
Nitric oxide	1·0390	·2315	·165	·2406	1·403
Carbon monoxide	0·9674	·2450	·173	·2370	1·416
Carbon dioxide	1·5290	·2163	·171	·3307	1·265
Sulphur dioxide	2·2470	·1544	·121	·3414	
Hydrochloric acid	1·2474	1845	·131	·2333	
Sulph. hydrogen	1·1912	·2431	·182	·2857	
Ammonia	·5894	·5083	·391	·2996	1·37
Marsh gas	·5527	·5929	·467	·3277	
Olefiant gas	·9672	·4040	·353	·4106	1·144
Water vapour	·6210	·4805	·369	·2984	
Carbon disulphide vapour	2·6325	·1570	·131	·4140	1·302
Ether vapour	2·5563	·4810	·454	1·2296	1·059
Mercury vapour	6·9783				1·667

* "Watt's Dictionary," Volume III. 1865, page 43. Mercury vapour, from Kundt, in *Pogg. Ann.*, 1876, page 353.

The table "Fusing Constants," &c., page 185, means that if we take, for example, a uniform copper wire of 1 cm. diameter, a current of 2530 amperes will raise it to 1054°C., and, therefore, fuse it; and if we take any conductor of copper of similar form, but of different diameter (d), the fusing current (C) is $C=2530d^{\frac{3}{2}}$. The fusing current of any other material is obtained from the equation $C=ad^{\frac{3}{2}}$.

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PLATE 1.

Fig. 1

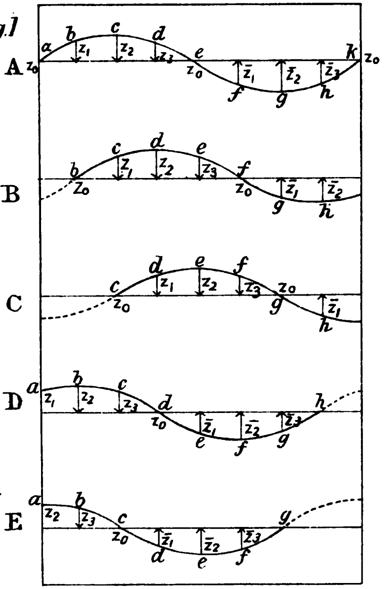


Fig. 2

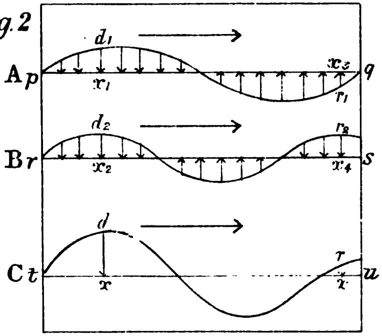
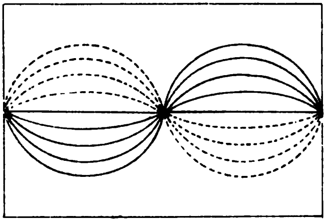


Fig. 5



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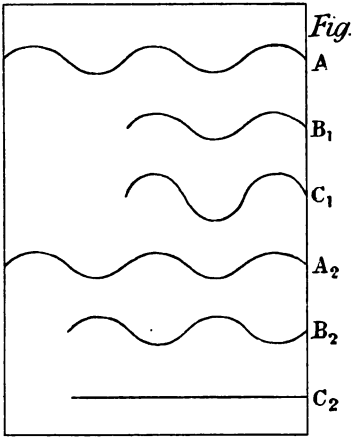


Fig. 4

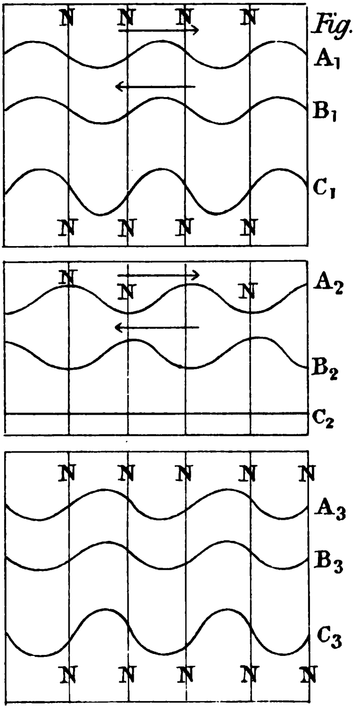
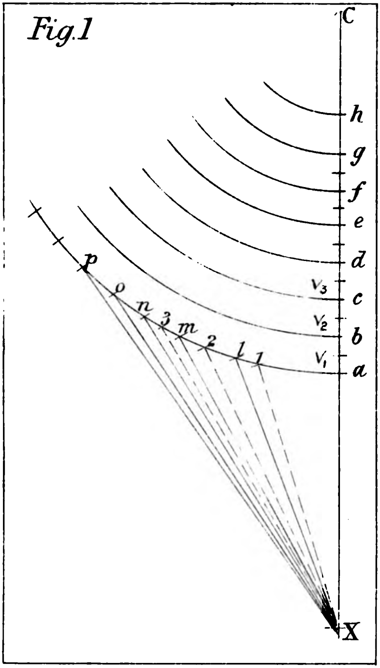


PLATE 2

Fig. 1



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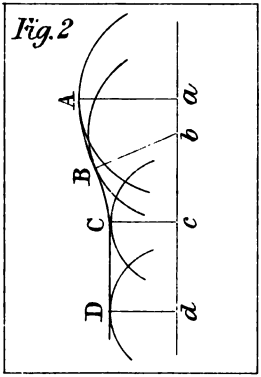

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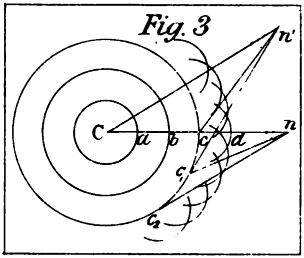

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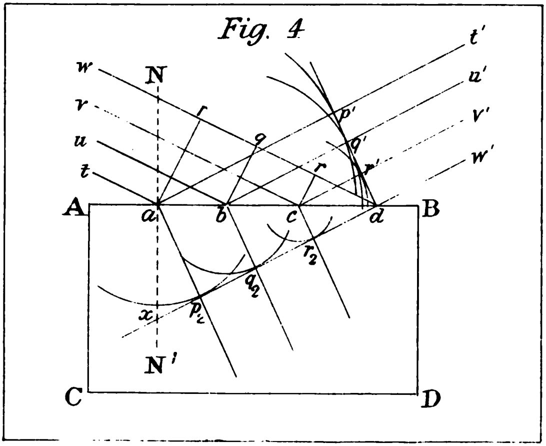

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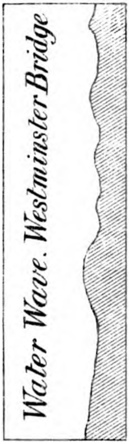

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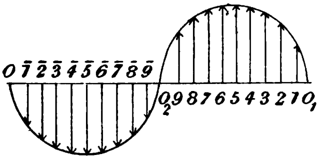

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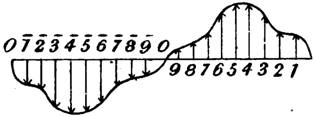

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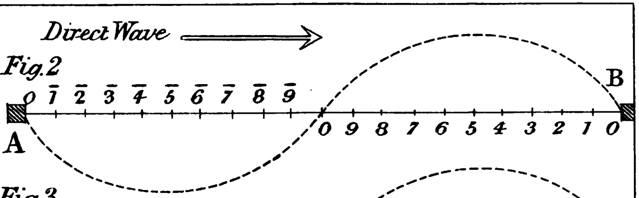

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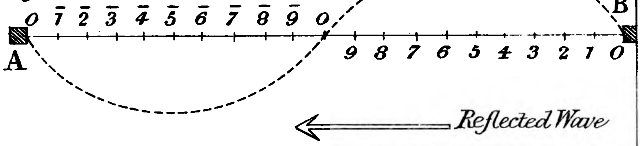

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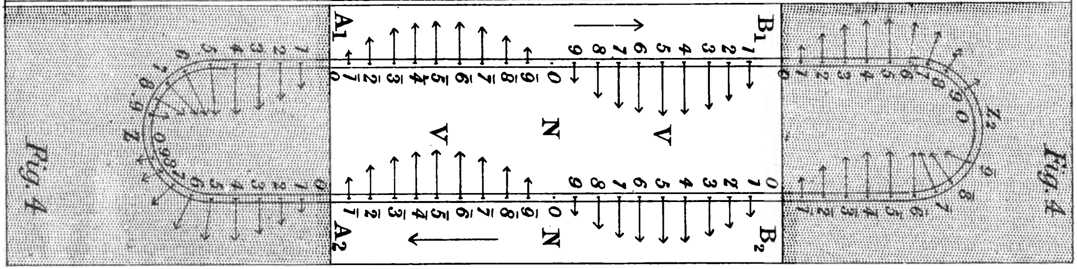
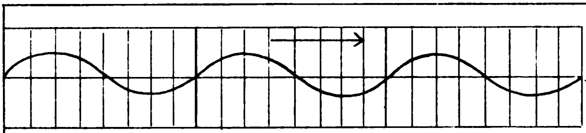


Fig. 4

Fig. 4

PLATE 4.



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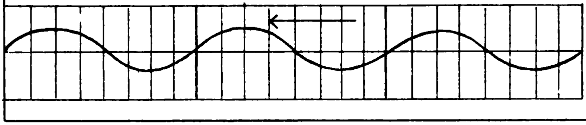

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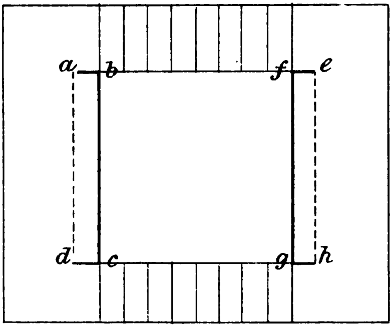

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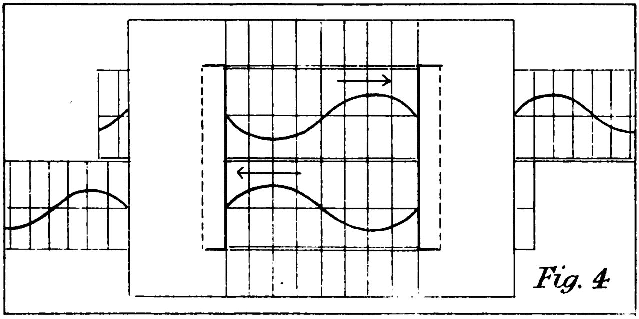

Fig. 4

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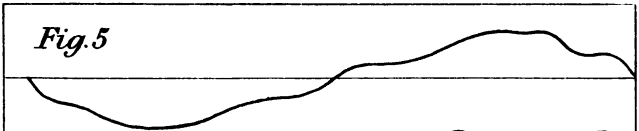
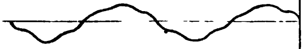


Fig. 6



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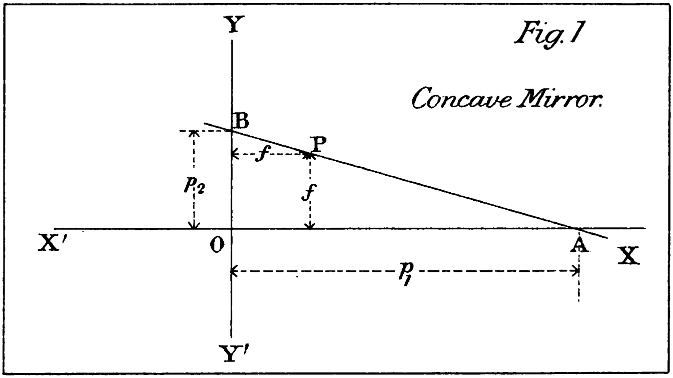


Fig. 1

Concave Mirror.

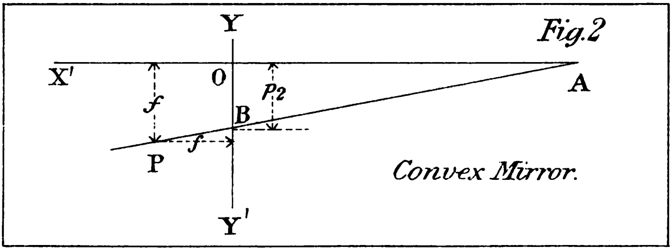


Fig. 2

Convex Mirror.

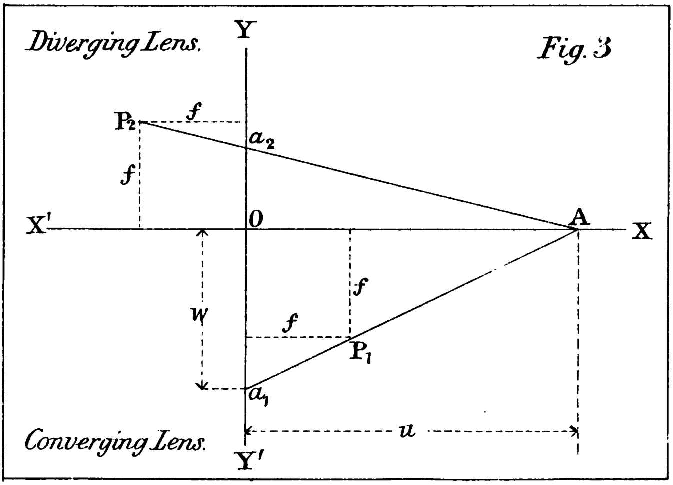
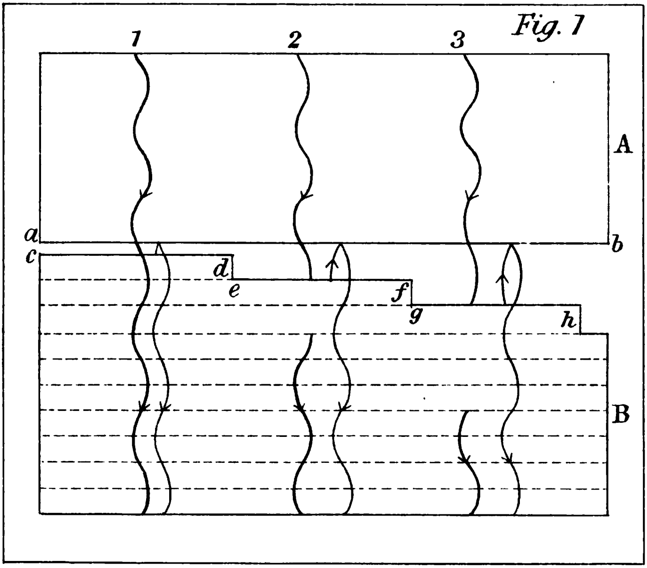


Fig. 3

Diverging Lens.

Converging Lens.

PLATE 6



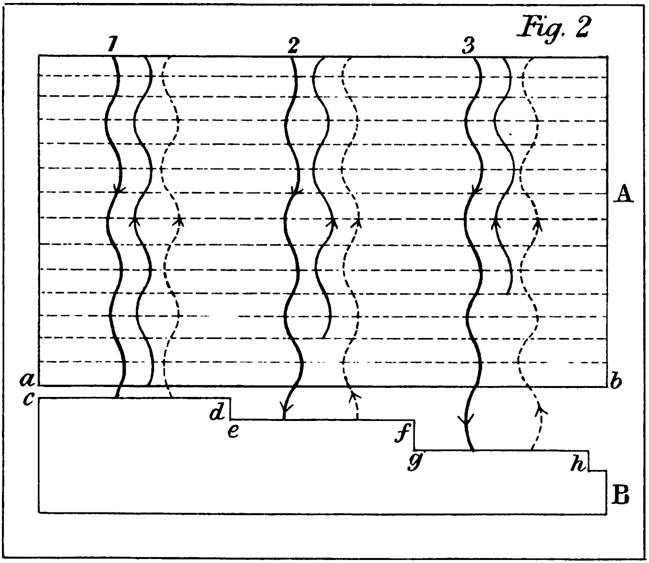
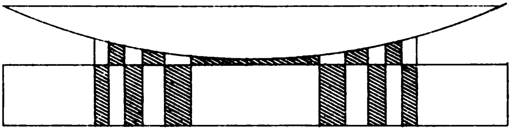


PLATE 7.

Fig. 1



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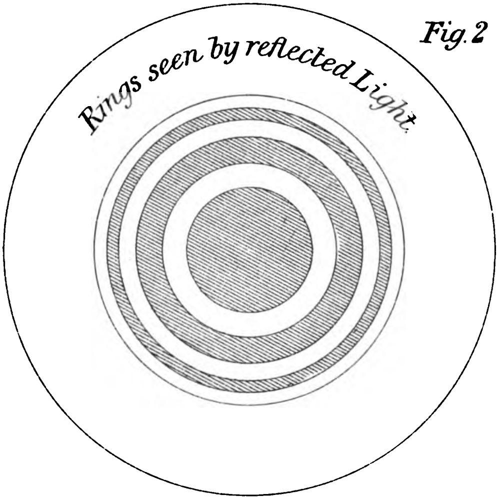

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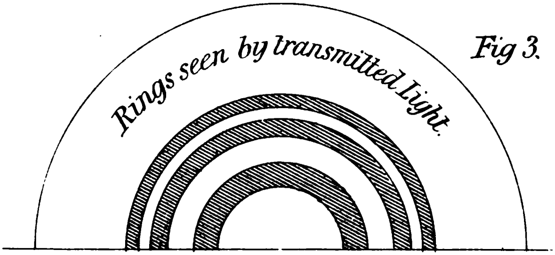


PLATE 8.

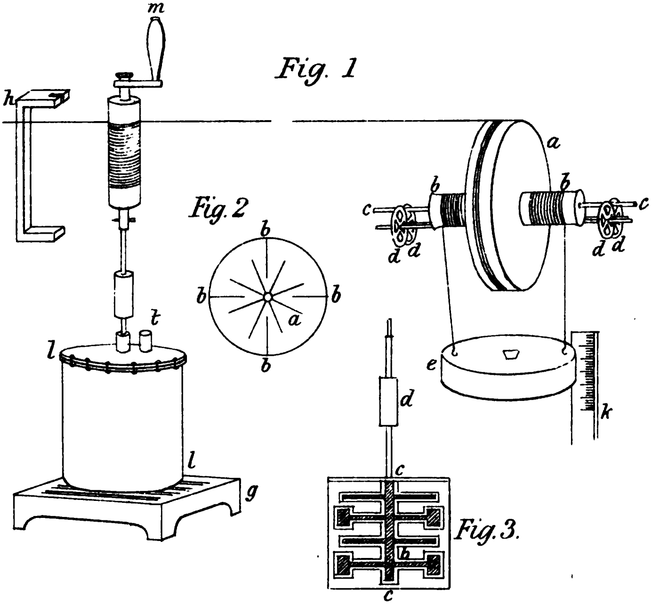
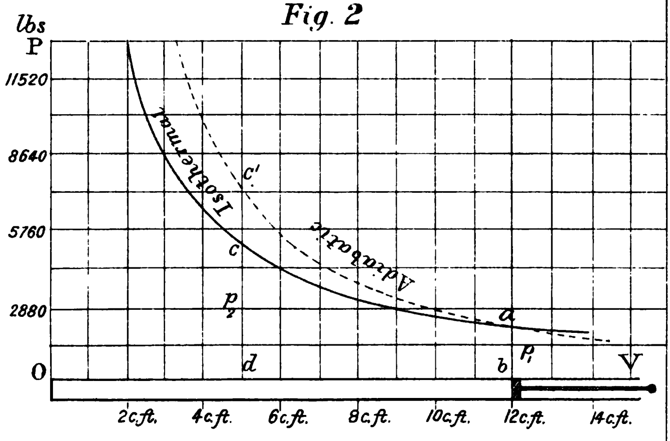


Fig. 1

Fig. 2

Fig. 3.

Fig. 2



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Dash below, letter below. Dash above, letter above.

N.	L. 0	1	2	3	4	5	6	7	8	9	
180	25 527	551	57$\overline{5}$	600	624	648	672	696	720	744	*a*
1	768	792	816	840	864	888	91$\underline{2}$	935	959	983	
2	26 007	031	055	07$\underline{9}$	102	126	150	174	19$\underline{8}$	221	
3	245	269	29$\underline{3}$	316	340	36$\underline{4}$	387	411	43$\underline{5}$	458	
4	48$\underline{2}$	505	529	55$\underline{3}$	$\overline{5}$76	60$\underline{0}$	623	64$\underline{7}$	670	69$\underline{4}$	*a*
5	26 71$\overline{7}$	741	76$\overline{4}$	788	811	83$\overline{4}$	858	88$\overline{1}$	905	928	*b*
6	95$\overline{1}$	975	998	*02$\overline{1}$	*045	*068	*091	*11$\overline{4}$	*138	*161	
7	27 184	20$\overline{7}$	231	254	277	300	323	34$\overline{6}$	370	393	
8	416	439	462	485	508	531	554	577	600	623	
9	646	669	692	715	738	761	784	807	830	85$\underline{2}$	
190	27 875	898	921	944	96$\underline{7}$	989	*012	*035	*058	*08$\underline{1}$	
1	28 103	126	14$\underline{9}$	171	194	217	24$\underline{0}$	262	28$\underline{5}$	307	
2	330	35$\underline{3}$	375	398	42$\underline{1}$	443	46$\underline{6}$	488	51$\underline{1}$	533	*b*
3	556	57$\overline{8}$	601	62$\overline{3}$	646	66$\overline{8}$	691	713	73$\overline{5}$	758	*c*
4	78$\overline{0}$	803	825	84$\overline{7}$	870	892	91$\overline{4}$	937	959	981	
5	29 00$\overline{3}$	026	048	070	09$\overline{2}$	115	137	159	181	20$\overline{3}$	
6	226	248	270	292	314	336	358	38$\overline{0}$	403	425	
7	447	469	491	513	535	557	579	601	623	645	
8	66$\underline{7}$	688	710	732	754	776	798	820	84$\underline{2}$	863	
9	885	907	929	951	97$\underline{3}$	994	*016	*038	*06$\underline{0}$	*081	
200	30 103	12$\underline{5}$	146	168	19$\underline{0}$	211	233	25$\underline{5}$	276	298	
1	32$\underline{0}$	341	36$\underline{3}$	384	406	42$\underline{8}$	449	47$\underline{1}$	492	51$\underline{4}$	
2	535	55$\underline{7}$	578	60$\underline{0}$	621	64$\underline{3}$	66$\underline{4}$	685	70$\underline{7}$	728	*c*
3	750	771	79$\overline{2}$	814	835	85$\overline{6}$	878	899	92$\overline{0}$	942	*d*
4	963	98$\overline{4}$	*006	*027	*048	*06$\overline{9}$	*091	*112	*133	*154	
5	31 17$\overline{5}$	197	218	239	260	281	302	32$\overline{3}$	345	366	
6	387	408	429	450	471	492	513	534	555	576	
7	597	618	639	660	681	702	723	744	76$\underline{5}$	785	
8	806	827	848	869	890	91$\underline{1}$	931	952	973	994	
9	32 01$\underline{5}$	035	056	077	09$\underline{8}$	118	139	160	18$\underline{1}$	201	*d*
N.	L. 0	1	2	3	4	5	6	7	8	9	

a	2	5	7	10	12	14	17	20	22
b	2	5	7	9	11	14	16	18	21
c	2	4	7	9	11	13	15	18	20
d	2	4	6	8	10	13	15	17	19
e	2	4	6	8	10	12	14	16	18

150

·176

The tables have a lateral index, the space on which this is printed being cut away in the book.

University of London Questions.

BY C J. WOODWARD, B.Sc.

◆

THIS work, published by permission, consists of the whole of the Questions in Mechanics and Experimental Science of the Matriculation Examination from the year 1864 to 1888 inclusive, classified under the different headings of the Syllabus. Answers are given, and with the more difficult questions hints are added as to the mode of working.

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Ang.	Sine.	Cosine.	Tang.
70°	0·940	0·342	2·747
71	·946	·326	·904
72	·951	·309	3·078
73	·956	·292	·271
74	·961	·276	·487
75°	0·968	0·259	3·732
76	·974	·242	4·011
77	·970	·225	·331
78	·976	·208	·705
79	·982	·191	5·145

Ang.	Sine.	Cosine.	Tang.
80°	0·985	0·174	5·671
81	·988	·156	6·314
82	·990	·139	7·115
83	·993	·122	8·144
84	·995	·105	9·514
85°	0·996	0·087	11·430
86	·998	·070	14·301
87	·999	·052	19·081
88	·999	·035	28·636
89	1·000	·017	57·290
90	1·000	0·000	∞

To find from
degrees and *mi*

Example.—
that sine 30°=
these. Now th
sents 60′; hen
30° 43′=0·500-

Cosine 30°=

By above re
30° 43′=0·866

The tangent
43/60=0·577+

Having given

Example.—
From table si
sents 60′. Hen
/0·014=60 × ·0

Example.—
while cosine 39

0·772--0·766
60 × 6/11 or 32′

R 1 1894
JUN 11 1894
JAN 27 1898
DEC 18 1899
JAN 30 1900
OC 21 [illegible]
APR 9 1910
JUL 7 1921

Note.—With this table angles and their functions can only be fo
where a difference of a minute or two is unimportant it will be fo

3 2044 081 578 528